数理解析研究所講究録
第 1783 巻 2012 年 77-125

Pseudoconvex domains in the Hopf surface.

by

Norman Levenberg and Hiroshi Yamaguchi

1 Introduction

Let $a \in \mathbb{C}\setminus\{0\}$ with $|a|>1$ and let $\mathcal{H}_a$ be the *Hopf manifold with respect to* a, i.e., $\mathcal{H}_a=\mathbb{C}^n\setminus\{0\}/\sim$ where $z'\sim z$ if and only if there exists $n\in\mathbb{Z}$ such that $z'=a^nz$ in $\mathbb{C}^n\setminus\{0\}$. In a previous paper [1] we showed that any pseudoconvex domain $D\subset\mathcal{H}_a$ with $C^\omega-$smooth boundary which is not Stein is biholomorphic to $T_a\times D_0$ where D_0 is a Stein domain in $\mathbb{P}^{n-1}$ with $C^\omega-$smooth boundary and T_a is a one-dimensional torus.

For $a,b\in\mathbb{C}\setminus\{0\}$ with $|b|\geq|a|>1$ we let $\mathcal{H}_{(a,b)}$ be the *Hopf surface with respect to* (a,b), i.e., $\mathcal{H}_{(a,b)}=\mathbb{C}^2\setminus\{(0,0)\}/\sim$, where $(z,w)\sim(z',w')$ if and only if there exists $n\in\mathbb{Z}$ such that $z'=a^nz,\ w'=b^nw$. We set

$$\rho=\frac{\log|b|}{\log|a|}\geq 1. \tag{1.1}$$

We remark that $\mathcal{H}_{(a,b)}$ is not a complex Lie group; however, with the aid of the technique of variation of domains in a complex Lie group developed in [1], we can characterize the domains with $C^\omega-$smooth boundary in $\mathcal{H}_{(a,b)}$ which are not Stein.

Theorem 1.1. *A pseudoconvex domain D with $C^\omega-$smooth boundary in $\mathcal{H}_{(a,b)}$ which is not Stein reduces to one of the following:*

Case **a**: *ρ is irrational.*

(**a**1) *There exist positive numbers $k_1<k_2$ such that*

$$D=\{k_1|z|^\rho<|w|<k_2|z|^\rho\}/\sim.$$

(**a**2) *There exists a positive number k such that*

$$D=\{|w|<k|z|^\rho\}/\sim.$$

(a3) *There exists a positive number k such that*

$$D = \{|w| > k|z|^{\rho}\}/\sim .$$

Each $|w| = k|z|^{\rho}$, $k > 0$ is biholomorphic to $|w| = |z|^{\rho}$ in $\mathcal{H}$.

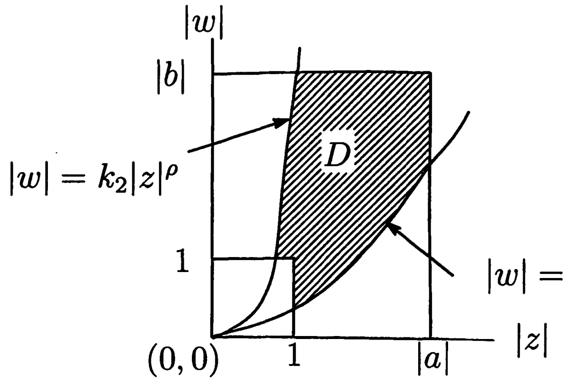
Case **b**: *$\rho = q/p$ is rational where $q > p \geq 1$ and $(p, q) = 1$. Setting*

$$\tau := \frac{1}{2\pi}\left(\frac{q}{p}\arg a - \arg b\right), \quad 0 \leq \arg a,\ \arg b < 2\pi, \tag{1.2}$$

we have:

(b1) *If τ is irrational, then D is of the form* **(a1)**, **(a2)** *or* **(a3)**.

(b2) *If $\tau = m/l$ is rational with $l \geq 1$ and $(l, m) = \pm 1$ or $\tau = 0$ (and we set $l = 1$):*
There exists a domain δ in $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ such that $D = \bigcup_{c\in\delta} \sigma_c$ where $\sigma_c := \{w = cz^{\rho}\}/\sim$ is the integral curve $[z_0, w_0]\exp tX_u$ with $c = w_0/z_0^{\rho} \neq 0, \infty$ of $X_u := (\log|a|)\, z\frac{\partial}{\partial z} + (\log|b|)\, w\frac{\partial}{\partial w}$. Also, σ_c is a compact curve which is equivalent to the one dimensional torus $T_{a^{l/p}} (= T_{b^{l/q}})$. If $c = 0$, then $\sigma_c = [z_0, 0]\exp tX_u = T_a \times \{0\}$ where $z_0 \neq 0$. If $c = \infty$, then $\sigma_c = [0, w_0]\exp tX_u = \{0\} \times T_b$ where $w_0 \neq 0$.



In the next section, we briefly discuss properties of the Hopf surface $\mathcal{H}_{(a,b)}$, and in section 3 we state without proof some preliminary results, including a classification of the holomorphic vector fields on $\mathcal{H}_{(a,b)}$ and their integral curves. We also indicate why the domains listed in Theorem 1.1 are not Stein. The proof of Theorem 1.1 is given in section 4 while the proofs of the results in section 3 are given at the end of the paper in Appendix A and Appendix B.

We would like to thank Professor Tetsuo Ueda for suggesting this problem.

2 Properties of the Hopf surface $\mathcal{H}_{(a,b)}$.

We write $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ and $(\mathbb{C}^2)^* := \mathbb{C}^2 \setminus \{(0,0)\}$. Fix $a, b \in \mathbb{C}^*$ with $1 < |a| \leq |b|$. For $(z,w), (z',w') \in (\mathbb{C}^2)^*$, we define the equivalence relation

$$(z,w) \sim (z',w') \quad \text{iff} \quad \exists\, n \in \mathbb{Z} \text{ such that } z' = a^n z, \ w' = b^n w.$$

The space $(\mathbb{C}^2)^*/\sim$ consisting of all equivalence classes

$$[z,w] := \{(a^n z, b^n w) : n \in \mathbb{Z}\}, \quad (z,w) \in (\mathbb{C}^2)^*$$

is called the *Hopf surface* $\mathcal{H} = \mathcal{H}_{(a,b)}$; it is a complex two-dimensional compact manifold.

For $z,\ z' \in \mathbb{C}^*$ we define $z \sim_a z'$ if and only if there exists $n \in \mathbb{Z}$ such that $z' = a^n z$ in $\mathbb{C}^*_z$. Then

$$T_a := \mathbb{C}^*/\sim_a \ \text{ and } T_b := \mathbb{C}^*/\sim_b$$

are complex one-dimensional tori, and $\mathcal{H}$ contains two disjoint compact analytic curves $\mathbf{T}_a = T_a \times \{0\}$ and $\mathbf{T}_b = \{0\} \times T_b$. We have $\mathbf{T}_a \cup \mathbf{T}_b = \{(z,w) \in (\mathbb{C}^2)^* : zw = 0\}/\sim$ in $\mathcal{H}$; for simplicity we write $\mathbf{T}_a \cup \mathbf{T}_b = \{zw = 0\}$. We consider the subdomain $\mathcal{H}^*$ of $\mathcal{H}$ defined by

$$\mathcal{H}^* := \mathcal{H} \setminus \{zw = 0\}. \tag{2.1}$$

Thus $\mathcal{H}$ is a compactification of $\mathcal{H}^*$ by two disjoint one-dimensional tori. The set $\mathcal{H}^*$ is a complex Lie group and will play a crucial role in this work.

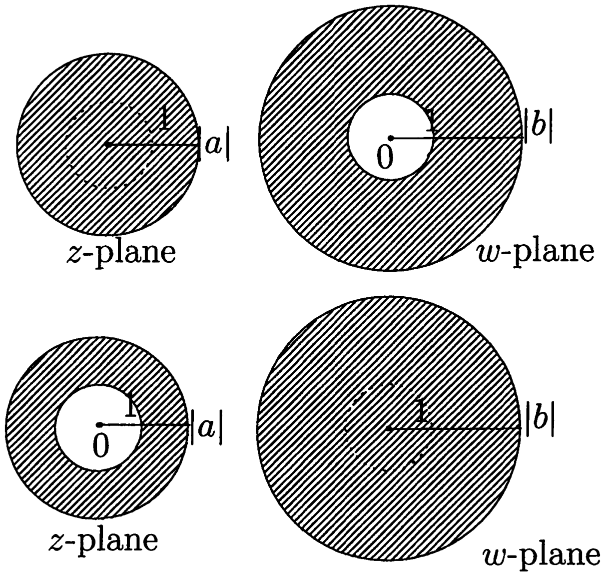
We give a more precise description of the Hopf surface. A fundamental domain for $\mathcal{H}$ is

$$\begin{aligned}\mathcal{F} :&= (\{|z| \leq |a|\} \times \{|w| \leq |b|\}) \setminus (\{|z| \leq 1\} \times \{|w| \leq 1\}) \\ &= E_1 \cup E_2 \Subset (\mathbb{C}^2)^*,\end{aligned} \tag{2.2}$$

where

$$\begin{aligned}E_1 &= E_1' \times E_1'' := \{|z| \leq |a|\} \times \{1 \leq |w| \leq |b|\}, \\ E_2 &= E_2' \times E_2'' := \{1 \leq |z| \leq |a|\} \times \{|w| \leq |b|\}.\end{aligned}$$

For $k = 0, \pm 1, \ldots$ we set $\mathcal{F}_k := \mathcal{F} \times (a^k, b^k)$. Then $\mathcal{F}_0 = \mathcal{F}$ and each $\mathcal{F}_k$ is a fundamental domain.



The Hopf surface $\mathcal{H}$ is obtained by gluing the boundaries of $\partial\mathcal{F}$ in the following way: setting

$$\begin{aligned} L'_a &:= \{|z| \le |a|\} \times \{|w| = |b|\}, \quad L'_1 = \{|z| \le 1\} \times \{|w| = 1\}; \\ L''_b &:= \{|z| = |a|\} \times \{|w| \le |b|\}, \quad L''_1 = \{|z| = 1\} \times \{|w| \le 1\}, \end{aligned}$$

we have the identifications:

$$\begin{aligned} &(1)\ \ (z, w) \in L'_a \quad \text{with} \quad (z/a, w/b) \in L'_1; \\ &(2)\ \ (z, w) \in L''_b \quad \text{with} \quad (z/a, w/b) \in L''_1. \end{aligned}$$

We set

$$\mathcal{I} = \{(a^n, b^n) \in \mathbb{C}^* \times \mathbb{C}^* : n \in \mathbb{Z}\} \subset \mathbb{C}^* \times \mathbb{C}^*, \tag{2.3}$$

which is a discrete set in $\mathbb{C}^* \times \mathbb{C}^*$.

For a set $D \subset \mathcal{H}$ we will often simply describe D as a set of points in $(\mathbb{C}^2)^*$ where the equivalence relation $\sim$ is understood. If there is possibility of confusion we will write

$$\widetilde{D} = \{(z, w) \in (\mathbb{C}^2)^* : [z, w] \in D\} \subset (\mathbb{C}^2)^*,$$

so that $\widetilde{D} = \widetilde{D} \times \mathcal{I}$ and hence $\widetilde{D}/\sim\, = D$.

As an example, which will also illustrate the difference between the Lie group $\mathcal{H}^*$ and the Hopf surface $\mathcal{H}$, let $D = \mathbb{C}_z \times \{w\}$ where $w \neq 0$. As a subset of $\mathcal{H}^*$, the complex curve $D \cap (\mathbb{C}^* \times \mathbb{C}^*)/\sim$ is not relatively compact and is equivalent to $\mathbb{C}^*$. As a complex curve in $\mathcal{H}$, $D/\sim$ is not closed and is equivalent to $\mathbb{C}$. Moreover, if $|b|^{k-1} < |w| < |b|^k$, then $(0, w) \in \mathcal{F}_k$ and

$$D/\sim \,= D_0 \cup D_1 \cup D_2 \cup \cdots$$

where

$$D_0 = \{|z| < |a|^k\} \times \{w\}, \quad D_n = \{|a|^{k-1} \leq |z| \leq |a|^k\} \times \{w/b^n\}, \quad n = 1, 2, \ldots$$

Thus D_0 is a disk and D_n, $n = 1, 2, \ldots$ are annuli such that $D_{n+1} = D_n \times (1, 1/b)$, $n = 1, 2, \ldots$. Hence D_n, $n = 1, 2, 3, \ldots$ are conformally equivalent, and $\lim_{n\to\infty} D_n = \mathbf{T}_a$ in $\mathcal{H}$.

Following T. Ueda, we consider the following real-valued function $U[z, w]$ on $\mathcal{H}^*$:

$$U[z, w] = \frac{\log|z|}{\log|a|} - \frac{\log|w|}{\log|b|} \quad \text{for} \quad [z, w] \in \mathcal{H}^*.$$

This has the following properties:

(1) $U[z, w]$ is a pluriharmonic function on $\mathcal{H}^*$ satisfying

$$\lim_{[z,w]\to\mathbf{T}_a} U[z, w] = -\infty \qquad \text{and} \qquad \lim_{[z,w]\to\mathbf{T}_b} U[z, w] = +\infty,$$

thus for any interval $I \Subset (-\infty, \infty)$, the subdomain $U^{-1}(I)$ of $\mathcal{H}^*$ is relatively compact in $\mathcal{H}^*$.

(2) $|U[z, w]| := \mathrm{Max}\{U[z, w], -U[z, w]\}$ is a plurisubharmonic exhaustion function for $\mathcal{H}^*$ which is pluriharmonic everywhere except on the Levi-flat set

$$\frac{\log|z|}{\log|a|} = \frac{\log|w|}{\log|b|}, \quad \textit{i.e.,} \quad |w| = |z|^\rho \quad \text{in} \quad \mathcal{H}^*.$$

(3) For $c \in (-\infty, +\infty)$, the level set

$$S_c: \quad U[z, w] = c$$

is equal to $|w| = k|z|^\rho$ where $k = e^{-c\log|b|} > 0$. Thus $\{k_1|z|^\rho \leq |w| \leq k_2|z|^\rho\}$ is equal to $U^{-1}([c_1, c_2])$ where $k_i = e^{-c_i \log|b|}$; while $\{|w| \leq k|z|^\rho\}$ is equal to $U^{-1}([c, +\infty)) \cup \mathbf{T}_a$; and $\{|w| \geq k|z|^\rho\}$ is equal to $U^{-1}((-\infty, c])) \cup \mathbf{T}_b$ where $k = e^{-c\log|b|}$.

From (2) and (3), each of the domains D in (**a**1), (**a**2) and (**a**3) in the statement of Theorem 1.1 contains a compact, Levi-flat hypersurface S_c for appropriate c; hence each such D is not Stein. In case (**b**2), D contains a compact torus σ_c and we will have the same conclusion.

3 Preliminary results

In this section, we discuss two basic results which we will need. The first concerns holomorphic vector fields in $\mathcal{H}$, while the second concerns pseudoconvex domains with C^ω–smooth boundary in $\mathbb{C}^2$. We consider the linear space of all holomorphic vector fields X of the form

$$X = \alpha z \frac{\partial}{\partial z} + \beta w \frac{\partial}{\partial w}, \quad \alpha, \beta \in \mathbb{C}$$

in $\mathbb{C}^* \times \mathbb{C}^*$. Any such X clearly induces a holomorphic vector field on $\mathcal{H} = \mathcal{H}_{(a,b)}$. The integral curve of X with initial value $(z_0, w_0) \in (\mathbb{C}^2)^*$ is

$$(z_0, w_0) \exp tX = \begin{cases} z = z_0 e^{\alpha t}, \\ w = w_0 e^{\beta t}. \end{cases} \qquad t \in \mathbb{C}.$$

Therefore,

$$\left(\frac{z}{z_0}\right)^\beta = \left(\frac{w}{w_0}\right)^\alpha, \quad \text{hence} \quad w = \left(\frac{w_0}{z_0^{\beta/\alpha}}\right) z^{\beta/\alpha}.$$

In particular, we consider

$$X_u := (\log|a|)\, z \frac{\partial}{\partial z} + (\log|b|)\, w \frac{\partial}{\partial w}. \tag{3.1}$$

The integral curve of X_u with initial value $(1,1)$ is

$$\exp tX_u = \begin{cases} z = e^{(\log|a|)t}, \\ w = e^{(\log|b|)t}. \end{cases} \qquad t \in \mathbb{C}.$$

Thus $w = z^\rho$. We set $\tilde{\sigma}_u := \{\exp tX_u : t \in \mathbb{C}\} \subset \mathcal{H}^*$ and denote by $\tilde{\Sigma}_u$ the closure of $\tilde{\sigma}_u$ in $\mathcal{H}$.

The next lemma gives more precise information about the integral curves and will be crucial in the proof of Lemma 4.2. For notational purposes, for rational $\rho = \frac{\log|b|}{\log|a|}$, we write $\rho = q/p,\ p \geq 1,\ (p,q) = 1$ and we define $\tau := ((q/p)\arg a - \arg b)/2\pi$. For τ rational, if $\tau = 0$, we define $l = 1$; if $\tau \neq 0$, we define l by $\tau = m/l,\ l \geq 1,\ (m,l) = \pm 1$.

Lemma 3.1. *1. For $X_u = (\log|a|)\, z\,\frac{\partial}{\partial z} + (\log|b|)\, w\,\frac{\partial}{\partial w}$ we have:*

(1) In case ρ is irrational or τ is irrational, $\tilde{\Sigma}_u = \{|w| = |z|^\rho\}$ is a real three-dimensional Levi-flat closed hypersurface in $\mathcal{H}^*$ with $\tilde{\Sigma}_u \Subset \mathcal{H}^*$.

(2) If τ is rational, then $\tilde{\sigma}_u \sim T_{a^{l/p}} (= T_{b^{l/q}})$ as Riemann surfaces.

2. For $X = \alpha z\frac{\partial}{\partial z} + \beta w\frac{\partial}{\partial w} \notin \{cX_u : c \in \mathbb{C}\}$, the integral curve $\sigma := \{\exp tX : t \in \mathbb{C}\}$ in $\mathcal{H}^*$ is not relatively compact in $\mathcal{H}^*$. If we let Σ denote the closure of σ in $\mathcal{H}$, then:

(1) If $\alpha, \beta \neq 0$, we have $\Sigma \supset \mathbf{T}_a \cup \mathbf{T}_b$.

(2) If only one of α or β is not 0, e.g., $\alpha \neq 0$ and $\beta = 0$, we have $\Sigma \supset \mathbf{T}_a$ and $\Sigma \cap \mathbf{T}_b = \emptyset$.

The proof of Lemma 3.1 is in Appendix A.

We now turn to an elementary property of a pseudoconvex domain D with C^ω–smooth ∂D in $\mathbb{C}^2$. In $\mathbb{C}^2 = \mathbb{C}_z \times \mathbb{C}_w$ we consider disks

$$\Delta_1 = \{|z| < r_1\}, \quad \Delta_2 = \{|w| < r_2\}$$

and the bidisk $\Delta = \Delta_1 \times \Delta_2$. Let D be a pseudoconvex domain with C^ω boundary in Δ. We do not assume D is relatively compact. Thus there exists a C^ω–smooth, real-valued function $\psi(z,w)$ on $\overline{\Delta}$ such that

$$D = \{(z,w) \in \Delta : \psi(z,w) < 0\};$$
$$\partial D \cap \Delta = \{(z,w) \in \Delta : \psi(z,w) = 0\},$$

and on $\psi(z,w) = 0$ we have both $\nabla_{(z,w)}\psi(z,w) \neq 0$ and the Levi form $\mathcal{L}\psi(z,w) \geq 0$. We write out this last condition:

$$\begin{aligned}\mathcal{L}\psi(z,w) &= \frac{\partial^2\psi}{\partial z\partial\overline{z}}\left|\frac{\partial\psi}{\partial w}\right|^2 - 2\Re\left\{\frac{\partial^2\psi}{\partial z\partial\overline{w}}\frac{\partial\psi}{\partial\overline{z}}\frac{\partial\psi}{\partial w}\right\} + \frac{\partial^2\psi}{\partial w\partial\overline{w}}\left|\frac{\partial\psi}{\partial z}\right|^2 \\ &\geq 0 \qquad \text{on } \psi(z,w) = 0. \end{aligned} \tag{3.2}$$

We may assume

$$\psi(0,0) = 0 \quad \text{and} \quad \frac{\partial \psi}{\partial w}(0,0) \neq 0$$

so that $\psi(0,w) = 0$ is a C^ω−smooth simple arc in Δ_2 passing through $w = 0$.

We set $\mathcal{S} := \partial D \cap \Delta$,

$$D(z) := \{w \in \Delta_2 : (z,w) \in D\} \subset \Delta_2; \text{ and}$$
$$S(z) := \{w \in \Delta_2 : (z,w) \in \mathcal{S}\} \subset \Delta_2,$$

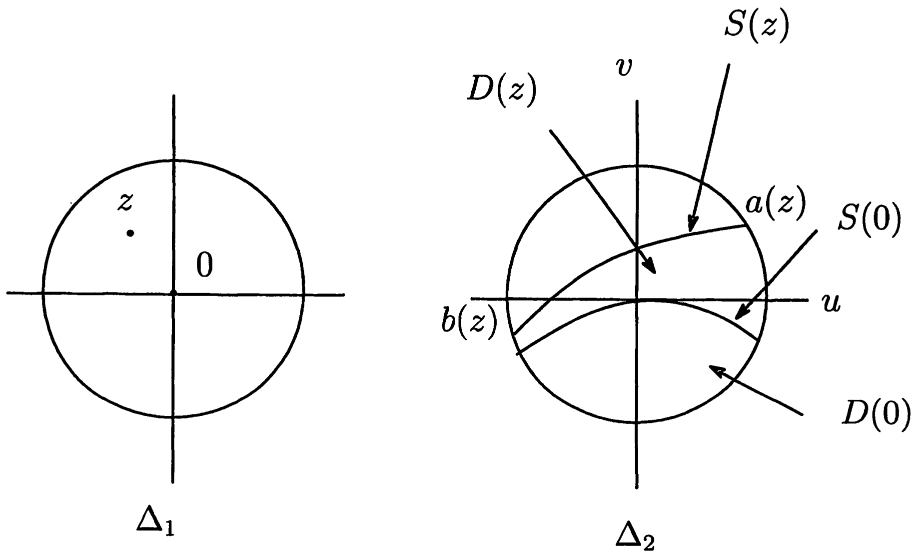
so that $D = \cup_{z\in\Delta_1}(z, D(z)) \subset \Delta$ and $\mathcal{S} = \cup_{z\in\Delta_1}(z, S(z)) \subset \Delta$. Taking $r_1, r_2 > 0$ sufficiently small we can insure that

(i) for each $z \in \Delta_1$, $D(z)$ is a non-empty domain in Δ_2 and $S(z)$ is a C^ω−smooth open arc in Δ_2 connecting two points $a(z)$ and $b(z)$ on $\partial\Delta_2$;

(ii) $0 \in S(0)$.

We also need the following condition:

$\boxed{(iii)}$ $\psi(z,0) \not\equiv 0$ in Δ_1, hence, for any disk $\delta_1 = \{|z| < r\} \subset \Delta_1$, there exists $z_0 \in \delta_1$ with $0 \notin S(z_0)$.



Under these conditions we have the following.

Lemma 3.2. *For any disk* $\delta_1 = \{|z| < r\} \subset \Delta_1$, *there exists a disk* $\delta_2 = \{|w| < r'\} \subset \Delta_2$ *with*

$$\bigcup_{z \in \delta_1} S(z) \supset D(0) \cap \delta_2.$$

The proof of Lemma 3.2 is in Appendix B. This result will be used in proving Lemma 4.1.

4 Construction of the plurisubharmonic exhaustion function $-\lambda[z, w]$ on D

Let $(\alpha, \beta) \in \mathbb{C}^* \times \mathbb{C}^*$. If we define

$$(\alpha, \beta) : \ [z, w] \in \mathcal{H} \mapsto [\alpha z, \beta w] \in \mathcal{H},$$

then (α, β) is an automorphism of $\mathcal{H}$. Thus $\mathbb{C}^* \times \mathbb{C}^*$ acts as a commutative group of automorphisms of $\mathcal{H}$ with identity element $e = (1, 1)$. Although $\mathbb{C}^* \times \mathbb{C}^*$ is not transitive on $\mathcal{H}$, it is transitive on $\mathcal{H}^*$. Hence $\mathcal{H}^*$ *is a homogeneous space with Lie transformation group* $\mathbb{C}^* \times \mathbb{C}^*$. For any $[z, w] \in \mathcal{H}^*$ the isotropy subgroup $I_{[z,w]}$ of $\mathbb{C}^* \times \mathbb{C}^*$ is

$$\begin{aligned} I_{[z,w]} :&= \{(\alpha, \beta) \in \mathbb{C}^* \times \mathbb{C}^* : (\alpha, \beta)[z, w] = [z, w]\} \\ &= \{(a^n, b^n) \in \mathbb{C}^* \times \mathbb{C}^* : n \in \mathbb{Z}\} \\ &= \mathcal{I} \ \text{ in } (2.3), \end{aligned}$$

and thus is independent of $[z, w] \in \mathcal{H}^*$. We have

$$\mathcal{H}^* = (\mathbb{C}^* \times \mathbb{C}^*)/\mathcal{I}.$$

In what follows we consider the restriction to $\mathbb{C}^* \times \mathbb{C}^*$ of the Euclidean metric $ds^2 = |dz|^2 + |dw|^2$ on $\mathbb{C}^2$, and we fix a positive real-valued function $c(z, w)$ of class C^ω on $\mathbb{C}^2$.

In this section we always assume that $D \subset \mathcal{H}$ *is a pseudoconvex domain with* C^ω*–smooth boundary in* $\mathcal{H}$. We note, as observed at the end of section 2, that

$$\text{if } D \Supset \mathbf{T}_a \text{ or } D \Supset \mathbf{T}_b, \text{ then } D \text{ is not Stein.} \tag{4.1}$$

We define

$$D^* := D \cap \{zw \neq 0\} \subset \mathcal{H}^*$$

(see (2.1)). The distinction between $D \subset \mathcal{H}$ and $D^* \subset \mathcal{H}^*$ will be very important. Since $(z, w) \in \mathbb{C}^* \times \mathbb{C}^*$ defines an automorphism of $\mathcal{H}$, for $[z, w] \in D$ we can define

$$D[z, w] = \{(\alpha, \beta) \in \mathbb{C}^* \times \mathbb{C}^* : (\alpha, \beta)[z, w] \in D\} \subset \mathbb{C}^* \times \mathbb{C}^*.$$

Equivalently, using the notation $D \cap \mathbf{T}_a = [D_a, 0]$ and $D \cap \mathbf{T}_b = [0, D_b]$,

$$D[z,w] = \left((\frac{1}{z}, \frac{1}{w}) \cdot D \right) \times \mathcal{I} \qquad \text{if } [z,w] \in D^*;$$

$$D[z,0] = \big(\frac{1}{z} D_a, \mathbb{C}^*\big) \times \mathcal{I} \qquad \text{if } [z,0] \in D \cap \mathbf{T}_a;$$

$$D[0,w] = \big(\mathbb{C}^*, \frac{1}{w} D_b\big) \times \mathcal{I} \qquad \text{if } [0,w] \in D \cap \mathbf{T}_b.$$

We note the following:

(1) $D[e] = \widetilde{D} \setminus \{zw = 0\} = \widetilde{D^*}$; $[z, w] \in D$ if and only if $e \in D[z, w]$;

(2) For each $[z, w] \in D$, $D[z, w]$ is an open set with C^ω boundary $\partial D[z, w]$ but it is not relatively compact in $\mathbb{C}^* \times \mathbb{C}^*$. We have

  (i) $D[z, w] = D[z, w] \times \mathcal{I}$;

  (ii) For $[z, w] \in D^*$ we define

$$D^*[z, w] = \{(\alpha, \beta) \in \mathbb{C}^* \times \mathbb{C}^* : (\alpha, \beta)[z, w] \in D^*\}.$$

  Then $D[z, w] = D^*[z, w]$.

(3) (i) For $[z, w] \in D^*$ we have

$$D[z, w] = \widetilde{D^*} \times (\frac{1}{z}, \frac{1}{w}) \,, \tag{4.2}$$

and for $[z, w], [z', w'] \in D^*$

$$D[z', w'] = (\frac{z}{z'}, \frac{w}{w'}) D[z, w]. \tag{4.3}$$

In particular, the sets $D[z, w]$ for $[z, w] \in D^*$ are biholomophic in $\mathbb{C}^* \times \mathbb{C}^*$.

(ii) For any two points $[z,0],[z',0] \in D \cap \mathbf{T}_a$

$$D[z',0] = (\frac{z}{z'},1)D[z,0].$$

In particular, the sets $D[z,0]$ for $[z,0] \in D^*$ are biholomophic in $\mathbb{C}^* \times \mathbb{C}^*$.

(3) Fix $[z_0,0] \in D \cap \mathbf{T}_a$ and let $[z_n,w_n] \in D^*$ $(n=1,2,\ldots)$ with $[z_n,w_n] \to [z_0,0]$ as $n \to \infty$ in $\mathcal{H}$. For $0<r<R$, consider the product of annuli

$$\mathcal{A}(r,R) : \{r<|z|<R\} \times \{r<|w|<R\} \subset \mathbb{C}^* \times \mathbb{C}^*.$$

Then

$$\lim_{n\to\infty} \partial D[z_n,w_n] \cap \mathcal{A}(r,R) = \partial D[z_0,0] \cap \mathcal{A}(r,R) \tag{4.4}$$

in the Hausdorff metric as compact sets in $\mathbb{C}^* \times \mathbb{C}^*$.

We set

$$\mathcal{D} := \bigcup_{[z,w]\in D} ([z,w], D[z,w]). \tag{4.5}$$

This is a pseudoconvex domain in $D \times (\mathbb{C}^* \times \mathbb{C}^*)$ which we consider as a function-theoretic "parallel" variation

$$\mathcal{D} : [z,w] \in D \to D[z,w] \subset \mathbb{C}^* \times \mathbb{C}^*.$$

Since $e \in D[z,w]$ for $[z,w] \in D$, we have the c-Green function $g([z,w],(\xi,\eta))$ with pole at e and the c-Robin constant $\lambda[z,w]$ for $(D[z,w],e)$ with respect to the metric ds^2 on $\mathbb{C}^* \times \mathbb{C}^*$ and the function $c(z,w)>0$. We call $[z,w] \to \lambda[z,w]$ the *c-Robin function for D.*

We have the following fundamental result.

Lemma 4.1.

1. *$-\lambda[z,w]$ is a plurisubharmonic function on D.*

2. *We have the following:*

   (a) For any $[z_0,w_0] \in \partial D^*$, $\lim_{[z,w]\to[z_0,w_0]} \lambda[z,w] = -\infty$.

(b) If $\emptyset \neq \partial D \cap \mathbf{T}_a \neq \mathbf{T}_a$ *then for any* $[z_0, 0] \in \partial D \cap \mathbf{T}_a$ *we have* $\lim_{[z,w]\to[z_0,0]} \lambda[z,w] = -\infty$ *(and similarly if* $\mathbf{T}_a$ *is replaced by* $\mathbf{T}_b$*).*

3. If $\partial D \not\supset \mathbf{T}_a$ *and* $\partial D \not\supset \mathbf{T}_b$*, then* $-\lambda[z,w]$ *is a plurisubharmonic exhaustion function for* D*.*

Proof. Note that 3. follows from 1. and 2. We divide the proof of 1. into two steps.

1st step. $-\lambda[z,w]$ *is pseudoconvex on* D^*.

Fix $[\zeta_0] = [z_0, w_0] \in D^*$. Let $\mathbf{a} \in \mathbb{C}^2 \setminus \{0\}$ with $\|\mathbf{a}\| = 1$ and let $B = \{|t| < r\} \subset \mathbb{C}_t$ be a small disk such that the complex line $l : t \in B \to [\zeta(t)] = [z(t), w(t)] = [\zeta_0] + \mathbf{a}t$ passing through $[\zeta_0]$ is contained in D^*. It suffices to prove that $-\lambda(t) := -\lambda[z(t), w(t)]$ is subharmonic on B, i.e.,

$$\frac{\partial^2 \lambda(t)}{\partial t \partial \bar{t}} \leq 0 \quad \text{on } B.$$

For brevity we write

$$D(t) : = D[\zeta(t)] \subset \mathbb{C}^* \times \mathbb{C}^* \text{ for } t \in B;$$
$$g(t, (z,w)) : = g([\zeta(t)], (z,w)) \quad \text{for } (z,w) \in D[\zeta(t)].$$

By (4.3) we have

$$D(t) = D(0)\ [\zeta(t)]^{-1} = D[\zeta_0]\ (\frac{1}{z(t)}, \frac{1}{w(t)}) \quad \text{in } \mathbb{C}^* \times \mathbb{C}^*. \tag{4.6}$$

We thus have the parallel variation of domains $D(t)$ in $\mathbb{C}^* \times \mathbb{C}^*$ with parameter $t \in B$:

$$\mathcal{D}|_B : t \in B \to D(t) \subset \mathbb{C}^* \times \mathbb{C}^*.$$

We write

$$\mathcal{D}|_B := \bigcup_{t \in B} (t, D(t)); \quad \partial\mathcal{D}|_B = \bigcup_{t\in B} (t, \partial D(t)) \quad \text{in } B \times (\mathbb{C}^* \times \mathbb{C}^*),$$

where again we identify the variation with the total space $\mathcal{D}|_B$. By (4.5), $\mathcal{D}|_B$ is a pseudoconvex domain in $B \times (\mathbb{C}^* \times \mathbb{C}^*)$ such that $\partial\mathcal{D}|_B$ is C^ω smooth.

Using the notation $\zeta = (z, w) \in \mathbb{C}^* \times \mathbb{C}^*$ and $g(t,\zeta) = g(t,(z,w))$, we have the following variation formula from Theorem 3.1 of [1]:

$$(\star) \qquad \frac{\partial^2 \lambda(t)}{\partial t \partial \bar{t}} = - c_2 \int_{\partial D(t)} K_2(t,\zeta) \|\nabla_\zeta \, g(t,\zeta)\|^2 dS_\zeta$$
$$- 4c_2 \iint_{D(t)} \Big(\Big|\frac{\partial^2 g(t,\zeta)}{\partial \bar{t} \partial z}\Big|^2 + \Big|\frac{\partial^2 g(t,\zeta)}{\partial \bar{t} \partial w}\Big|^2 \Big) dV_\zeta$$
$$- 2c_2 \iint_{D(t)} c(\zeta) \Big|\frac{\partial g(t,\zeta)}{\partial t}\Big|^2 dV_\zeta .$$

Here c_2 is the surface area of the unit sphere in $\mathbb{C}^2$; dV_ζ is the Euclidean volume element in $\mathbb{C}^2$;

$$K_2(t,\zeta) = \mathcal{L}(t,\zeta)/\|\nabla_\zeta \psi(t,\zeta)\|^3$$

where $\mathcal{L}(t,\zeta)$ is the diagonal Levi form defined by

$$\mathcal{L}(t,\zeta) = \frac{\partial^2 \psi}{\partial t \partial \bar{t}} \|\nabla_\zeta \psi\|^2 - 2\Re \left\{ \frac{\partial \psi}{\partial t} \Big(\frac{\partial \psi}{\partial \bar{z}} \frac{\partial^2 \psi}{\partial \bar{t} \partial z} + \frac{\partial \psi}{\partial \bar{w}} \frac{\partial^2 \psi}{\partial \bar{t} \partial w} \Big) \right\} + \Big\|\frac{\partial^2 \psi}{\partial t}\Big\|^2 \Delta_\zeta \psi;$$

and $\psi(t,\zeta)$ is a defining function of $\mathcal{D}|_B$. The quantity $K_2(t,\zeta)$ is independent of the defining function $\psi(t,\zeta)$ (cf., Chapter 3 of [1]). Since $\mathcal{D}|_B$ is pseudoconvex in $B \times (\mathbb{C}^* \times \mathbb{C}^*)$, we can choose $\psi(t,\zeta)$ so that $K_2(t,\zeta) \geq 0$ on $\partial \mathcal{D}|_B$, and hence $\frac{\partial^2 \lambda(t)}{\partial t \partial \bar{t}} \leq 0$ on B, proving the first step.

Since $c(z,w) > 0$ in $\mathbb{C}^* \times \mathbb{C}^*$, the variation formula immediately implies the following rigidity result which will be useful later (cf., Lemma 4.1 of [1]).

Remark 4.1. *If* $\frac{\partial^2 \lambda}{\partial t \partial \bar{t}}(0) = 0$, *then* $\frac{\partial g}{\partial t}(0,(z,w)) \equiv 0$ *on* $D(0)$, *i.e.,*

$$\frac{\partial g([\zeta_0] + \mathbf{a}t, (z,w))}{\partial t}\Big|_{t=0} \equiv 0 \textit{ on } D[\zeta_0].$$

2^{nd} step. *Plurisubharmonic extension of* $-\lambda[z,w]$ *to* D.

We fix a point of $D \cap [(T_a \times \{0\}) \cup (\{0\} \times T_b)])$, e.g., $[z_0, 0]$ with $z_0 \neq 0$. Let $[z_n, w_n] \in D^*$ $(n = 1, 2, \ldots)$ with $[z_n, w_n] \to [z_0, 0]$ as $n \to \infty$. By (4.4)

$$\lim_{n\to\infty} (g([z_n,w_n],(\alpha,\beta)) - g([z_0,0],(\alpha,\beta))) = 0$$

uniformly for (α,β) in $K \Subset D[z_0,0] \subset \mathbb{C}^* \times \mathbb{C}^*$.

It follows that $\lim_{n\to\infty}\lambda[z_n,w_n] = \lambda[z_0,0]$, i.e., $\lambda[z,w]$ is continuous and finite at $[z_0,0]$. Hence $\lambda[z,w]$ is continuous and finite-valued on D. Since $D\cap\mathbf{T}_a$ is a complex line, it follows from the first step that $-\lambda[z,w]$ extends to be subharmonic from $D^*\cap\mathbf{T}_a$ to $D\cap\mathbf{T}_a$. Hence $-\lambda[z,w]$ extends to be plurisubharmonic on D. □

We divide the proof of 2. in two steps; the first step is 2 (a).

1^{st} step. *Fix* $[z',w']\in\partial D^*$. *If* $[z,w]\in D\to[z',w']$ *in* $\mathcal{H}$, *then* $\lambda[z,w]\to -\infty$.

Since $[z',w']\in\partial D^*$, we have $z'\neq 0$ and $w'\neq 0$. If $[z,w]\in D^*$ tends to $[z',w']$ in $\mathcal{H}$, then $\partial D[z,w]\subset\mathbb{C}^*\times\mathbb{C}^*$ tends to the single point e in the sense that if we define $d[z,w]=\mathrm{dist}(\partial D[z,w],e)>0$, where

$$\mathrm{dist}(\partial D[z,w],e) := \mathrm{Min}\,\{\sqrt{|\xi-1|^2+|\eta-1\}^2}:\ (\xi,\eta)\in\partial D[z,w]\},$$

then $d[z,w]\to 0$ as $[z,w]\to[z',w']$. Indeed, let $[z,w]\in D$ approach $[z',w']$ in $\mathcal{H}$. By slightly deforming the fundamental domain $\mathcal{F}\subset\mathbb{C}^*\times\mathbb{C}^*$ if necessary, we may assume $(z',w'),(z,w)\in\mathcal{F}$. Since

$$\partial D[z,w]=\{(\frac{\alpha}{z},\frac{\beta}{w})\in\mathbb{C}^*\times\mathbb{C}^*:\ [\alpha,\beta]\in\partial D\}$$

and $[z',w']\in\partial D^*$,

$$d[z,w]=\mathrm{dist}(\partial D[z,w],e)\leq\sqrt{|\frac{z'}{z}-1|^2+|\frac{w'}{w}-1|^2}$$

which clearly tends to 0 as $[z,w]\to[z',w']$. Since $\partial D[z,w]$ is a *smooth* real three-dimensional hypersurface, it follows by standard potential-theoretic arguments that $-\lambda[z,w]\to+\infty$. □

It remains to prove 2 (b). Thus we assume $\emptyset\neq\partial D\cap\mathbf{T}_a\neq\mathbf{T}_a$.

2^{nd} step. *Fix* $[z_0,w_0]\in\partial D\cap\{zw=0\}$. *Then*

$$\lim_{[z,w]\to[z_0,w_0],\ [z,w]\in D}\lambda[z,w]=-\infty.$$

For the proof of this step we require Lemma 3.2. Fix $p_0=[z_0,w_0]\in\partial D\cap\{zw=0\}$. We want to show

$$\lim_{[z,w]\to[z_0,w_0],\ [z,w]\in D}\lambda[z,w]=-\infty.$$

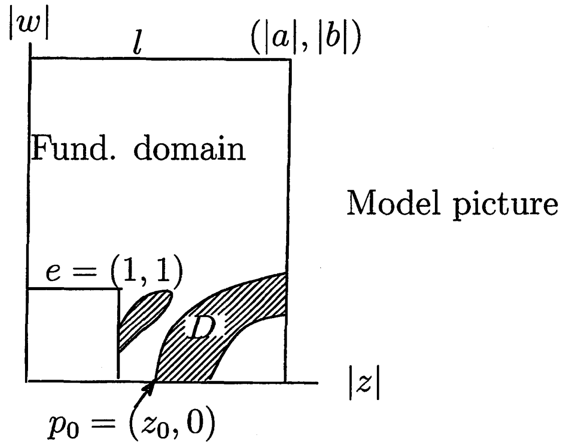
We may assume $p_0 = [z_0, 0]$ and we take a sequence $\{[z_n, w_n]\}_n \subset D$ which converges to p_0 in $\mathcal{H}$. We show

$$\lim_{n\to\infty} \lambda[z_n, w_n] = -\infty. \tag{4.7}$$

From continuity of $\lambda[z, w]$ in D, it suffices to prove (4.7) for $[z_n, w_n] \in D^*$. Moreover, since $\partial D[z_n, w_n]$ is smooth, as in the end of the first step, we need only show

$$\lim_{n\to\infty} \text{dist } (\partial D^*[z_n, w_n], e) = 0. \tag{4.8}$$

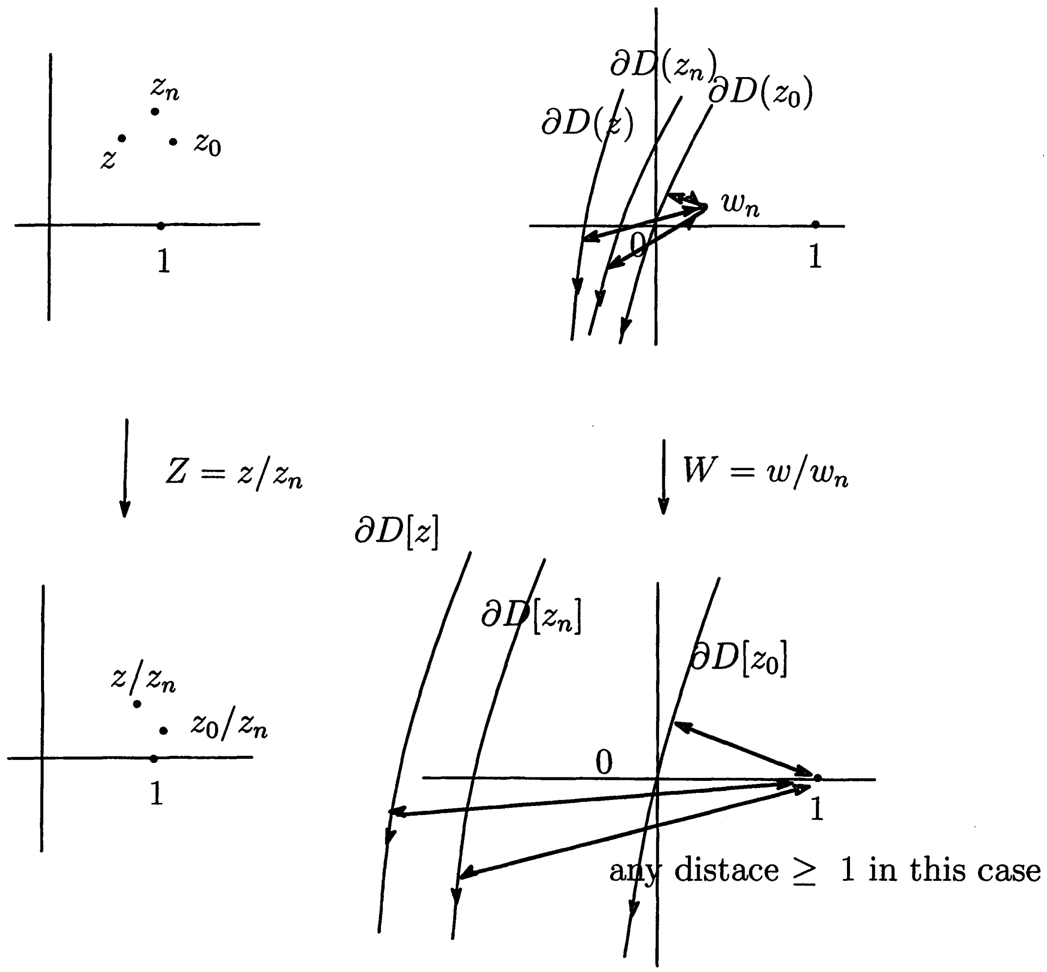
We digress to explain the subtlety of the problem.



.... [1]

[1]This is a "model" picture since, e.g., $(|a|, |b|)$ is a real two-dimensional torus while $(1, 1)$ is a point; similarly, the $|z|$-axis is a complex line $w = 0$, while l is a real three-dimensional surface.

Real picture of ∂D



Real picture of $\partial D[z_n, w_n]$

We write $D(z)$ for the slice of $D \subset \mathcal{H}$ over z and we write $D[z] \subset \mathbb{C}^*_w$ for the slice of $D[z,w] \subset \mathbb{C}^* \times \mathbb{C}^*$ over z. In this picture, $[z_n, w_n] \to p_0$ as $n \to \infty$ but we do not have the property dist $(\partial D[z_n, w_n], e) \to 0$. Thus (4.8) does not hold.

Using Lemma 3.2 and **pseudoconvexity** of the domain D in $\mathcal{H}$, we prove (4.8). We may assume that $p_0 = [z_0, 0] \in \partial D$ lies in the fundamental domain $\mathcal{F}$ and we take a sufficiently small bidisk $\Delta = \Delta_1 \times \Delta_2$ with center $(z_0, 0)$ so that $\Delta \subset \mathcal{F}$. Let $\psi(z,w)$ be a defining function of D in Δ, i.e., $\psi(z,w) \in C^\omega(\Delta)$ with $D \cap \Delta = \{\psi(z,w) < 0\}$ and $\partial D \cap \Delta = \{\psi(z,w) = 0\}$.

Since ∂D is smooth in $\mathcal{H}$, we have two cases:

$$\text{Case } (\mathbf{c1}): \quad \frac{\partial\psi}{\partial z} \not\equiv 0 \text{ on } \Delta; \qquad \text{Case } (\mathbf{c2}): \quad \frac{\partial\psi}{\partial w} \not\equiv 0 \text{ on } \Delta.$$

Apriori, we also have two cases relating to the behavior of $\psi(z,0)$ on Δ_1:

$$\text{Case } (\mathbf{d1}): \quad \psi(z,0) \not\equiv 0 \quad \text{on } \Delta_1; \quad \text{Case } (\mathbf{d2}): \quad \psi(z,0) \equiv 0 \quad \text{on } \Delta_1.$$

However, the hypothesis $\partial D \not\supset \mathbf{T}_a$ in 2 (b) and 3 of Lemma 4.1 together with the real-analyticity of ∂D imply that Case (d2) does not occur. Thus it suffices to prove (4.8) assuming that $\psi(z,0) \not\equiv 0$ on Δ_1.

Proof of (4.8) in Case (**c1**).

In this case, by taking a suitably smaller bidisk Δ if necessary, $l(0) := \{\psi(z,0) = 0\}$ is a C^ω−smooth arc in Δ_1 passing through $z = z_0$ and $l(0) \times \{0\} \subset \partial D \cap \Delta$. For $w \in \Delta_2$,

$$l(w) := \{z \in \Delta_1 : (z,w) \in \partial D \cap \Delta\}.$$

is a simple C^ω−smooth arc in Δ_1.

Fix $\varepsilon > 0$. Since $z_0 \neq 0$, we can find a disk $\delta_1 \subset \Delta_1$ with center z_0 such that

$$|\frac{z'}{z''} - 1| < \varepsilon \quad \text{for all } z', z'' \in \delta_1.$$

Now we take $\delta_2 : |w| < r < \varepsilon$ in Δ_2 so that each arc $l(w)$ passes through a certain point $\zeta(w)$ in δ_1. For sufficiently large n_0, if $n \geq n_0$ we have $(z_n, w_n) \in \delta_1 \times \delta_2$. Since $w_n \in \delta_2$, we have $\zeta(w_n) \in l(w_n) \cap \delta_1$ so that $(\zeta(w_n), w_n) \in \partial D$ in $\mathcal{H}$. Hence, $(\frac{\zeta(w_n)}{z_n}, \frac{w_n}{w_n}) = (\frac{\zeta(w_n)}{z_n}, 1) \in \partial D[z_n, w_n]$ in $\mathbb{C}^* \times \mathbb{C}^*$. Thus

$$\text{dist}\,(\partial D[z_n, w_n], e\,) \leq \text{dist}\,((\frac{\zeta(w_n)}{z_n}, 1), e\,) = |\frac{\zeta(w_n)}{z_n} - 1| < \varepsilon \quad \text{for for } n \geq n_0.$$

Proof of (4.8) in Case (**c2**).

In this case, by taking a suitably smaller bidisk Δ if necessary, $S(z_0) := \{\psi(z_0, w) = 0\}$ is a C^ω−smooth arc in Δ_1 passing through $w = 0$ and $\{z_0\} \times S(z_0) \subset \partial D \cap \Delta$. For $z \in \Delta_1$,

$$S(z) := \{w \in \Delta_2 : (z,w) \in \partial D \cap \Delta\},$$

is a simple $C^{\omega}-$smooth arc in Δ_2.

Fix $\delta_1 := \{|z-z_0| < r_1\} \Subset D(z_0)$. Case (d1) corresponds to the condition $\boxed{(iii)}$ in Lemma 3.2, thus this lemma implies that there exists a disk $\delta_2 := \{|w| < r_2\}$ such that

$$\cup_{z\in\delta_1} S(z) \supset D(z_0) \cap \delta_2. \tag{4.9}$$

Fix $\varepsilon > 0$. Taking r_1 sufficiently small, we can insure that

$$|\frac{z'}{z''} - 1| < \varepsilon \quad \text{for all } \ z', \ z'' \in \delta_1.$$

Take a disk $\delta_2 \subset \Delta_2$ satisfying (4.9). For sufficiently large n_0, if $n \geq n_0$ we have $(z_n, w_n) \in \delta_1 \times \delta_2$. We divide the points $w_n \in \delta_2$ into two types:

$$\text{Case (i)}: \ w_n \in \delta_2 \cap D(z_0); \quad \text{Case (ii)}: \ w_n \in \delta_2 \setminus D(z_0).$$

In Case (i), using (4.9) we can find $z^* \in \delta_1$ with $w_n \in S(z^*)$ so that $(z^*, w_n) \in \partial D$ in $\mathcal{H}$ (see w_n, $z^*, \partial D(z^*)$ in the figure below). Thus, $(\frac{z^*}{z_n}, \frac{w_n}{w_n}) = (\frac{z^*}{z_n}, 1)$ in $\partial D[z_n, w_n]$ in $\mathbb{C}^* \times \mathbb{C}^*$ and hence

$$\text{dist } (\partial D[z_n, w_n], e\ ) \leq \text{dist } ((\frac{z^*}{z_n}, 1), e\ ) = |\frac{z^*}{z_n} - 1| < \varepsilon \quad \text{for all } \ n \geq n_0.$$

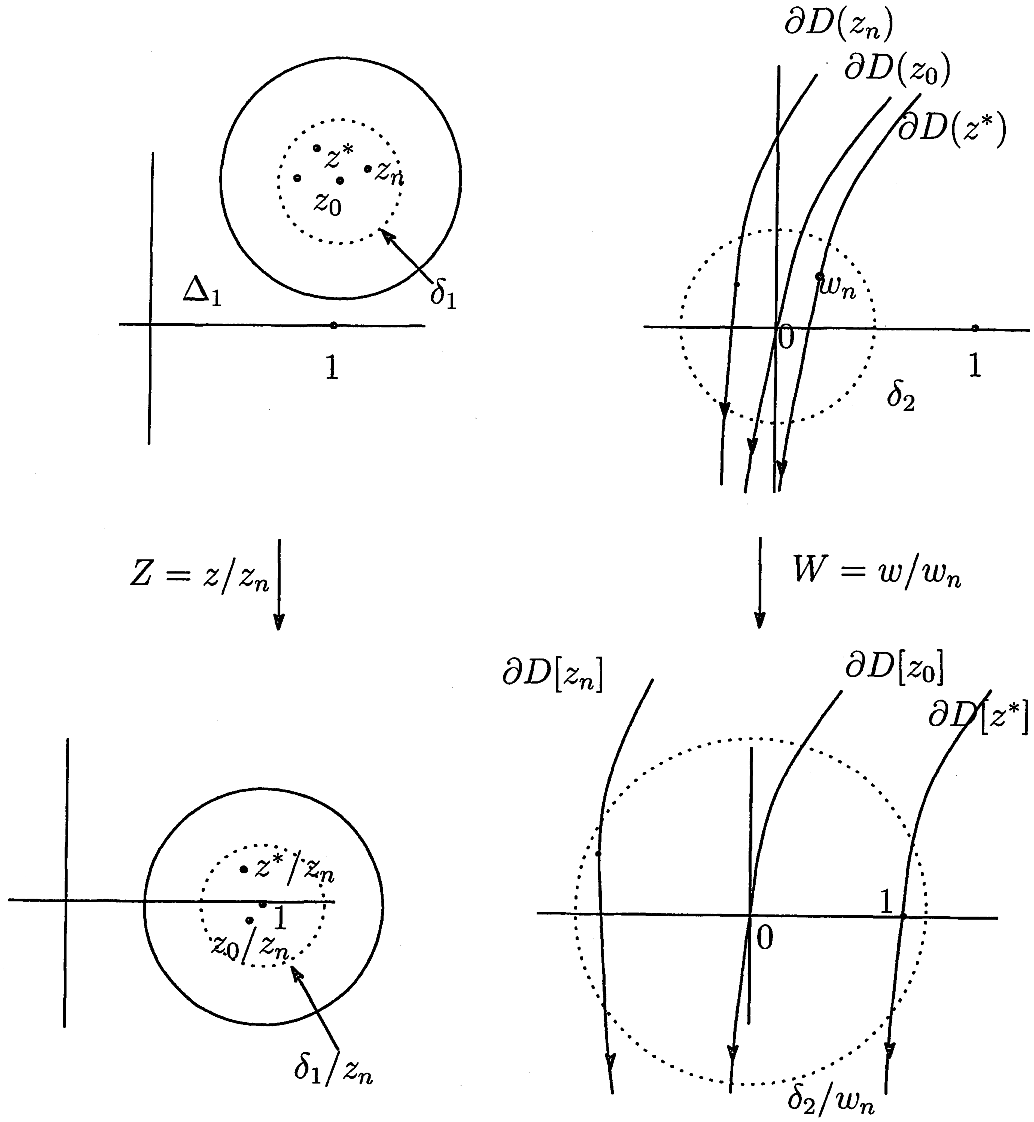
In Case (ii), let $\ell = [z_n, z_0]$ be a segment in δ_1. We can find $z^* \in \ell$ with $w_n \in \partial D(z^*)$. Indeed, as z goes from z_n to z_0 along ℓ, the arcs $\partial D(z) \cap \delta_2$ transform from $\partial D(z_n) \cap \delta_2$ to $\partial D(z_0) \cap \delta_2$ in a continuous fashion. Since $[z_n, w_n] \in D^*$, we can find $z^* \in \ell$ with $w_n \in \partial D(z^*)$.

Thus $(z^*, w_n) \in \partial D^*$, so that $(z^*/z_n, 1) \in \partial D^*[z_n, w_n]$, and hence

$$\text{dist } (\partial D[z_n, w_n], e\ ) \leq \text{dist } ((\frac{z^*}{z_n}, 1), e\ ) = |\frac{z^*}{z_n} - 1| < \varepsilon \quad \text{for all } \ n \geq n_0,$$

which is (4.8). This completes the proof of 2 (b) and 3 in Lemma 4.1. $\square$

Picture of the real three-dimensional set ∂D



Picture of the real three-dimensional set $\partial D[z_n, w_n]$.

We next relate the possible absence of strict plurisubharmonicity of the c-Robin function $\lambda[z,w]$ on a pseudoconvex domain D in $\mathcal{H}$ with existence of holomorphic vector fields on $\mathcal{H}$ with certain properties. This is in the spirit of, but does not follow from, Lemma 5.2 of [1]. Recall that in the case ρ is rational and τ is rational, we defined $\sigma_c := \{w = cz^\rho\}/\sim$ to be the integral curve $[z_0, w_0]\exp tX_u$ with $c = w_0/z_0^\rho \neq 0, \infty$ of $X_u := (\log|a|)\, z\,\frac{\partial}{\partial z} + (\log|b|)\, w\,\frac{\partial}{\partial w}$.

Lemma 4.2. *Let D be a pseudoconvex domain with C^ω-smooth boundary in $\mathcal{H}$ and let $\lambda[z,w]$ be the c-Robin function on D. Assume that there exists a point $p_0 = [z_0, w_0]$ in D^* at which $-\lambda[z,w]$ is not strictly plurisubharmonic.*

(1) There exists a holomorphic vector field $X = \alpha z\frac{\partial}{\partial z}dz + \beta w\frac{\partial}{\partial w}dw \neq 0$ on $\mathcal{H}$ such that if $[z,w] \in D^*$ (resp. ∂D^*), then the integral curve $I[z,w] := [z,w]\exp tX$ in $\mathcal{H}$ is contained in D^* (resp. ∂D^*).

(2) The form of the vector field X in (1) and the domain D are determined as follows:

(i) if $\partial D \not\supset \mathbf{T}_a$ and $\partial D \not\supset \mathbf{T}_b$, then $X = cX_u$ for some $c \neq 0$ with X_u in (3.1). If ρ is irrational or ρ is rational and τ is irrational, D is of type (**a1**) *in Theorem 1.1. If ρ is rational and τ is rational, D is of type* (**b2**) *in Theorem 1.1. In all cases, we have $\partial D \cap (\mathbf{T}_a \cup \mathbf{T}_b) = \emptyset$.*

(ii) if $\partial D \supset \mathbf{T}_a$ and $\partial D \not\supset \mathbf{T}_b$, then we have two cases:

(ii-a) $X = cX_u$ for some $c \neq 0$ and D is of type (**b2**). *We then have $D = \cup_{c\in\delta}\sigma_c$ where δ is a domain in $\mathbb{P}^1 = \mathbb{C}\cup\{\infty\}$ with smooth boundary $\partial\delta$ which contains 0 but not ∞.*

(ii-b) $X = cz\frac{\partial}{\partial z}$ for some $c \neq 0$. Then D is a domain of "Nemirovskii type": [2] *$b > 1$ and $D = \mathbb{C}_z^* \times \{Au + Bv < 0\}/\sim$, where $A, B \in \mathbb{R}$ with $(A,B) \neq (0,0)$ (here $w = u + iv$).*

(ii') if $\partial D \supset \mathbf{T}_b$ and $\partial D \not\supset \mathbf{T}_a$, we have the result analogous to (ii).

(iii) If $\partial D \supset \mathbf{T}_a \cup \mathbf{T}_b$, then D is of type (**b2**)*: $D = \cup_{c\in\delta}\sigma_c$ where δ is a domain in $\mathbb{P}^1$ with smooth boundary $\partial\delta$ with $0, \infty \in \partial\delta$, and $X = cX_u$ for some c.*

---

[2] *Nemiroviskii's theorem in [2]:* Let $a > 1$ and let $\mathcal{H} = \mathcal{H}_{a,a}$. Let $D = \mathbb{C}_z \times \{\Re w > 0\}/\sim \subset \mathcal{H}$. Then ∂D is Levi-flat and D is Stein.

Proof. Since $\lambda[z,w]$ is plurisubharmonic on D and is not strictly plurisubharmonic at $p_0 = [z_0, w_0] \in D^*$, we can find a holomorphic vector field $X = \alpha z\frac{\partial}{\partial z}dz + \beta w\frac{\partial}{\partial w}dw \neq 0$ on $\mathcal{H}$ such that

$$\frac{\partial^2 \lambda[\, p_0 \exp tX\,]}{\partial t \partial \bar{t}}\bigg|_{t=0} = 0. \tag{4.10}$$

We shall show that this X coincides with X in (1). Since $p_0 \in D^*$, we can take a small disk $B = \{|t| < r\}$ with $p_0 \exp tX \subset D^*$ for $t \in B$. We set $D(t) = D[p_0 \exp tX] \subset \mathbb{C}^* \times \mathbb{C}^*$ so that $D(0) = D[p_0]$. We let $g(t,(z,w))$ (resp. $\lambda(t)$) denote the c-Green function $g([p_0 \exp tX],(z,w))$ (resp. the c-Robin constant $\lambda[p_0 \exp tX]$) for $(D(t), e)$ and $t \in B$. We set $\mathcal{D}|_B = \cup_{t\in B}(t, D(t)) \subset B \times (\mathbb{C}^* \times \mathbb{C}^*)$ which we consider as the variation

$$\mathcal{D}|_B : \ t \in B \to D(t) = D[p_0 \exp tX] \subset \mathbb{C}^* \times \mathbb{C}^*.$$

By (4.3) we have

$$\begin{aligned} D(t) &= D[p_0 \exp tX] = D[[z_0, w_0] \exp tX] \\ &= D[z_0, w_0] \exp(-tX) = D[z_0, w_0]\big(e^{-\alpha t}, e^{-\beta t}\big) \quad \text{in } \mathbb{C}^* \times \mathbb{C}^*. \end{aligned}$$

Using the same reasoning as in the first step of the proof of Lemma 4.1 together with Remark 4.1 we see from (4.10) and the real analyticity of $\partial\mathcal{D}|_B = \cup_{t\in B}(t, \partial D(t))$ in $B \times (\mathbb{C}^* \times \mathbb{C}^*)$ that

$$\frac{\partial g(t,(z,w))}{\partial t}\bigg|_{t=0} = 0 \quad \text{on } D[z_0, w_0] \cup \partial D[z_0, w_0]. \tag{4.11}$$

For a fixed $t \in B$ we consider the automorphism

$$(Z, W) \to (z, w) = F(t, (Z, W))$$

of $\mathbb{C}^* \times \mathbb{C}^*$ where

$$F(t,(Z,W)) := (Z,W)\big(\frac{1}{z_0}, \frac{1}{w_0}\big) \exp(-tX) = \big(\frac{Ze^{-\alpha t}}{z_0}, \frac{We^{-\beta t}}{w_0}\big).$$

Then

$$(z,w) \to (Z,W) = F^{-1}(t,(Z,W)) = \Big(zz_0 e^{\alpha t}, ww_0 e^{\beta t}\Big).$$

By (4.2) we have

$$D(t) = \widetilde{D^*}\,(\frac{1}{z_0}, \frac{1}{w_0}) \exp(-tX) = \widetilde{D^*}\,(\frac{e^{-\alpha t}}{z_0}, \frac{e^{-\beta t}}{w_0}) \quad \text{in } \mathbb{C}^* \times \mathbb{C}^*,$$

so that $D(t) = F(t, \widetilde{D}^*)$. We note that $\widetilde{D^*} \subset \mathbb{C}^* \times \mathbb{C}^*$ is independent of $t \in B$. We set

$$G(t,(Z,W)) := g(t,(z,w)) \quad \text{where } (z,w) = F(t,(Z,W)), \quad (Z,W) \in \widetilde{D^*}.$$

Since

$$g(t,(z,w)) = G(t, F^{-1}(t,(z,w)) = G(t,(zz_0e^{\alpha t}, ww_0e^{\beta t})),$$

we have

$$\begin{aligned}
&\frac{\partial g}{\partial t}(t,(z,w)) \\
&= \frac{\partial G}{\partial t}(t,(Z,W)) + \frac{\partial G}{\partial Z}(t,(Z,W))\alpha z z_0 e^{\alpha t} + \frac{\partial G}{\partial W}(t,(Z,W))\beta w w_0 e^{\beta t} \\
&= \frac{\partial G}{\partial t}(t,(Z,W)) + \alpha Z\frac{\partial G}{\partial Z}(t,(Z,W)) + \beta W\frac{\partial G}{\partial W}(t,(Z,W))
\end{aligned}$$

where $(Z,W) = F^{-1}(t,(z,w))$. Since, for each $t \in B$,

$$G(t,(Z,W)) \equiv 0 \quad \text{on } \partial\widetilde{D^*}, \tag{4.12}$$

we have

$$\frac{\partial G}{\partial t}(t,(Z,W)) = 0 \quad \text{on } \partial\widetilde{D^*}.$$

It follows from (4.11) that

$$\alpha Z\frac{\partial G}{\partial Z}(t,(Z,W)) + \beta W\frac{\partial G}{\partial W}(t,(Z,W)) = 0 \quad \text{on } \partial\widetilde{D^*}.$$

Together with (4.12), this says that the holomorphic vector field

$$X = \alpha Z\frac{\partial}{\partial Z} + \beta W\frac{\partial}{\partial W}$$

on $\mathbb{C}^* \times \mathbb{C}^*$ is complex tangential on the boundary $\partial\widetilde{D^*}$. Hence X coincides with that in (4.10).

Thus, for any $(z,w) \in \partial\widetilde{D^*}$, the integral curve $(z,w)\exp tX \subset \partial\widetilde{D^*}$ for all $t \in \mathbb{C}$. It follows that for any $(z,w) \in \widetilde{D^*}$, the integral curve $(z,w)\exp tX$ is contained in $\widetilde{D^*}$:

$$\widetilde{D^*}\exp tX = \widetilde{D^*}, \text{ for all } t \in \mathbb{C}.$$

This implies

$$D[[z,w]\exp tX] = D[z,w] \subset \mathbb{C}^* \times \mathbb{C}^*, \text{ for all } t \in \mathbb{C} \tag{4.13}$$

if $[z,w] \in D^*$ since

$$D[[z,w]\exp tX] = \widetilde{D^*}\ (\frac{1}{z},\frac{1}{w})\ \exp(-tX) = \widetilde{D^*}\ (\frac{1}{z},\frac{1}{w}) = D[z,w].$$

But for $[z,w] \in D^*$ (*resp.* ∂D^*) it is clear that

$$[z,w]\exp tX \subset D^* \ (resp.\ \partial D^*) \quad \text{in } \mathcal{H}$$
$$\text{if and only if}$$
$$(z,w)\exp tX \subset \widetilde{D^*}\ (resp.\ \partial\widetilde{D^*}) \quad \text{in } \mathbb{C}^* \times \mathbb{C}^*,$$

which proves (1) of Lemma 4.2.

To prove assertion (2) we first observe by (4.13)

$$\lambda[z,w] = \lambda[[z,w]\exp tX], \text{ for all } t \in \mathbb{C}$$

for any $[z,w] \in D^*$. In case (2)(i) in Lemma 4.2, from 3 in Lemma 4.1, the Robin function $-\lambda[z,w]$ is an exhaustion function on D, and it follows that

$$\{[z,w]\exp tX : t \in \mathbb{C}\} \Subset D \text{ for } [z,w] \in D^*. \tag{4.14}$$

We also need the following conclusions from Lemma 3.1:

(α) If ρ in (1.1) is in case **a** or (**b1**) in Theorem 1.1, i.e., either ρ is irrational or ρ is rational and τ in (1.2) is irrational, then

$$\mathcal{H} = \left(\bigcup_{c\in(0,\infty)} \{|w| = c|z|^\rho : z \in \mathbb{C}^*\} \right) \cup (\mathbf{T}_a \cup \mathbf{T}_b)$$

and this is a disjoint union. Here $\Sigma_c := \{|w| = c|z|^\rho : z \in \mathbb{C}^*\}$ is the closure of the integral curve $\sigma[z_0,w_0] = [z_0,w_0]\exp tX_u$ with $c = |w_0/z_0^\rho|$ in $\mathcal{H}$; Σ_c is a real three-dimensional Levi flat hypersurface in $\mathcal{H}$ (and hence $\Sigma_c \Subset \mathcal{H}^*$). We set $\sigma_0 = \mathbf{T}_a$ and $\sigma_\infty = \mathbf{T}_b$ so that $\mathcal{H} = \bigcup_{c\in[0,\infty]} \sigma_c$.

(β) If ρ in (1.1) is in case (**b2**) in Theorem 1.1 so that τ in (1.2) is rational, then

$$\mathcal{H} = \left(\bigcup_{c \in \mathbb{C}^*} \{w = cz^\rho\} \right) \cup (\mathbf{T}_a \cup \mathbf{T}_b)$$

and this is a disjoint union. Here $\sigma_c := \{w = cz^\rho\}$ is the integral curve $[z_0, w_0] \exp tX_u$ with $c = w_0/z_0^\rho$; σ_c is a compact curve in $\mathcal{H}$ (and hence in $\mathcal{H}^*$) which is equivalent to the one-dimensional torus $T_{a^{l/p}}(= T_{b^{l/q}})$. We note that $\mathbf{T}_a = [z_0, 0] \exp tX_u$ where $z_0 \neq 0$ and $\mathbf{T}_b = [0, w_0] \exp tX_u$ where $w_0 \neq 0$. We set $\sigma_0 = \mathbf{T}_a$ and $\sigma_\infty = \mathbf{T}_b$ so that $\mathcal{H} = \bigcup_{c \in \mathbb{P}^1} \sigma_c$.

We now prove (2) (i). First we show that $X = cX_u$ for some $c \neq 0$. If not, i.e., if $X \not\in \{cX_u : c \in \mathbb{C}^*\}$, we take $[z, w] \in \partial D^*$ and let $\sigma = [z, w] \exp tX$ be the integral curve of X passing through $[z, w]$. From Lemma 3.1 part 2 (2), the closure Σ of σ in $\mathcal{H}$ contains $\mathbf{T}_a$ or $\mathbf{T}_b$, which contradicts the hypothesis $\partial D \not\supset \mathbf{T}_a$ and $\partial D \not\supset \mathbf{T}_b$ of (2) (i) in Lemma 4.2. Thus $X = cX_u$ for some $c \neq 0$.

By (4.14), for $[z, w] \in D^*$ the closure of the integral curve $I[z, w] := [z, w] \exp tX_u$ is compactly contained in D and hence lies in D^*. Using (α) and (β) it follows that

(α^*) $D^* = \bigcup_{c \in I} \{|w| = c|z|^\rho\}$, where I is an open interval in $(0, \infty)$; or

(β^*) $D^* = \bigcup_{c \in \delta} \{w = cz^\rho\}$, where δ is a domain in $\mathbb{C}^*$.

We next show that if $D \cap \mathbf{T}_a \neq \emptyset$ then $D \supset \mathbf{T}_a$, contradicting the hypothesis in (2) (i). We work in the case (α^*); the case (β^*) is similar. Thus let $[z_0, 0] \in D \cap \mathbf{T}_a$. Let U, V be sufficiently small disks such that

$$(z_0, 0) \in U \times V \Subset D \cap E_2$$

where recall $E_2 = \{1 \leq |z| \leq |a|\} \times \{|w| \leq |b|\} \subset \mathcal{F}$. Take $r_0 > 0$ with $(z_0, r_0) \in U \times V$. Note that $(z_0, tr_0) \in U \times V \Subset \widetilde{D}$ for all $0 \leq t \leq 1$. Define c_t by the relationship $tr_0 = c_t|z_0|^\rho$. Then for $0 < t < 1$, $\{|w| = c_t|z|^\rho\} \subset D$ by the properties of D; i.e.,

$$\{|w| = \frac{tr_0}{|z_0|^\rho}|z|^\rho\} \subset D \text{ for } 0 < t \leq 1.$$

Setting $R := \frac{tr_0}{|z_0|^\rho}$, we have

$$\bigcup_{c\in J}\{|w| = c|z|^\rho\} \subset D^* \text{ where } J = (0, R).$$

It follows that D contains the set

$$G := \{(z, w) \in E_2 : 1 < |z| < a,\ 0 < |w| < R\}.$$

Suppose $D \not\supset \mathbf{T}_a$. We use the pseudoconvexity of D to derive a contradiction. Observe that $D(0) := D \cap \mathbf{T}_a$ is a domain in $\mathbf{T}_a$ whose boundary l consists of smooth real one-dimensional curves. Fix $z' \in D(0)$ near l. Let $D(w)$ denote the slice of D corresponding to w for $0 < |w| < R$. We consider the Hartogs radius $r(w)$ for $D(w)$ centered at z'. Clearly $r(0) < r(w)$ for $0 < |w| < R$. Since $D \cap E_2$ is pseudoconvex in E_2, this contradicts the superharmonicity of $r(w)$. A completely similar argument shows that if $D \cap \mathbf{T}_b \neq \emptyset$ then $D \supset \mathbf{T}_b$. Thus either $D = D^*$ as in (α^*) or (β^*) or $D \setminus D^*$ consists of $\mathbf{T}_a$, $\mathbf{T}_b$, or $\mathbf{T}_a \cup \mathbf{T}_b$ with D^* as in (α^*) or (β^*). We verify that $D \setminus D^* = \mathbf{T}_a$ cannot happen; entirely similar proofs show that $D \setminus D^* = \mathbf{T}_b$ and $D \setminus D^* = \mathbf{T}_a \cup \mathbf{T}_b$ cannot occur. Indeed, if $D \setminus D^* = \mathbf{T}_a$, then $\partial D = \mathbf{T}_a$, which is a complex line. However, ∂D is assumed to be smooth; hence it must be a real three-dimensional surface. This completes the proof of (2) (i).

To prove (2) (ii), we note that under the condition $\partial D \supset \mathbf{T}_a$ and $\partial D \not\supset \mathbf{T}_b$, from Lemma 3.1 we have either $X = cX_u$ with $c \neq 0$ or $X = \alpha z\frac{\partial}{\partial z}$ with $\alpha \neq 0$. Using the same reasoning as in the proof of 2 (i) we conclude that D cannot be of the form **a** nor of the form (**b**1) in Theorem 1.1.

If $X = cX_u$ with $c \neq 0$, then D^* is of the form (β^*). Since $\partial D \supset \mathbf{T}_a$ and $\partial D \not\supset \mathbf{T}_b$ we arrive at the conclusion in (2) (ii-a). On the other hand, if $X = \alpha z\frac{\partial}{\partial z}$ with $\alpha \neq 0$, we first observe from the facts that $\partial D \supset \mathbf{T}_a$ and ∂D is $C^\omega-$smooth, for any $z_0 \in \mathbb{C}^*$ the slice of ∂D over $z = z_0$ is a C^ω curve $C(z_0) \subset \mathbb{C}_w$ passing through the origin $w = 0$. We can find a sufficiently small disk $V := \{|w| < r_0\}$ so that $C(z_0)$ divides V into two parts V' and V'' with $\{z_0\} \times V' \subset D$ and $\{z_0\} \times V'' \subset \overline{D}^c$. We set $\widetilde{C}(z_0) := C(z_0) \cap V$. By (1) in Lemma 4.2 we conclude that $\mathbb{C}^* \times V' \subset D$ and $\mathbb{C}^* \times V'' \subset \overline{D}^c$. Thus $\mathbb{C}^* \times \widetilde{C}(z_0) \subset \partial D$, which implies $\partial D \cap (\mathbb{C}^* \times V) = \mathbb{C}^* \times \widetilde{C}(z_0)$ and $D \cap (\mathbb{C}^* \times V) = \mathbb{C}^* \times V'$.

We use this geometric set-up to show that b must be a positive real number (hence $b > 1$). To see this, fix a point $w_0 \in \widetilde{C}(z_0)$ (resp. V') with $w_0 \neq 0$. Since $(z_0, w_0) \in \partial D$ (resp. V'), we have $\mathbb{C}^* \times \{w_0\} \subset \partial D$

(resp. D). In particular, $(a^n z_0, w_0) \in \partial D$ (resp. D) for any $n \in \mathbb{Z}$. Hence $(z_0, w_0/b^n) \in \partial D$ (resp. D) for any $n \in \mathbb{Z}$. Since $|b| > 1$ we can take N sufficiently large so that $w_0/b^N \in V$. It follows that $w_0/b^n \in \widetilde{C}(z_0)$ (resp. V') for any $n \geq N$.

We first show that b is real. If not, let $b = |b|e^{i\phi}$ where $|b| > 1$ and $0 < |\phi| < \pi$. We set $w_0 = |w_0|e^{i\varphi_0}$. Let $\mathbf{n}_0 = e^{i\theta_0}$ be a unit normal vector to $\widetilde{C}(z_0)$ at $w = 0$ pointing in to V''. Since $\widetilde{C}(z_0)$ is smooth, we can find r_1 sufficiently small with $0 < r_1 < r_0$ so that the sector $\mathbf{e} := \{re^{i\theta} : 0 < r < r_1,\ |\theta - \theta_0| < 2\pi/3\}$ is contained in V''. For any $N' \in \mathbb{Z}$, it is clear that there exists $n' > N'$ satisfying

$$|(\varphi_0 - n'\phi) - \theta_0| < 2\pi/3 \text{ modulo } 2\pi. \tag{4.15}$$

We take $N' > N$ so that $|w_0|/|b|^{N'} < r_1$, and then we choose $n' > N'$ with property (4.15). Then $w_0/b^{n'} \in \mathbf{e} \subset V''$, which contradicts the fact that $w_0/b^{n'} \in \widetilde{C}(z_0)$. Thus b is real.

We next show b is positive. If not, we have $b < -1$. We take $w_1 \in V' \setminus \{0\}$ close to 0. Then $(z, w_1) \in \widetilde{D}$ for all $z \in \mathbb{C}$. In particular, $(a^n z_0, w_1) \in \widetilde{D}$ for any $n \in \mathbb{Z}$; hence $(z_0, w_1/b^n) \in (\{z_0\} \times V) \cap D$ for n sufficiently large. In other words, for $n > N$ we have $w_1/b^n \in V'$. Since $b < -1$ it follows that $\{w_1/b^n : n \geq N\}$ lies on a line L passing through $w = 0$. Moreover, if we take a sufficiently small disk $V_0 := \{|w| < r_0\} \subset V$, then $L \cap V_0$ intersects the smooth curve $\widetilde{C}(z_0)$ transversally. At the point $z = 0$, $L \cap V_0$ divides into two segments L' and L'' with $L' = (L \cap V_0) \cap V'$ and $L'' = (L \cap V_0) \cap V''$. Since $b < -1$, for n sufficiently large, if $w_1/b^n \in L'$ then $w_1/b^{n+1} \in L''$. This contradicts the fact that $w_1/b^m \in V'$ for all m sufficiently large. Thus $b > 1$.

Consequently,

$$w \in \tilde{C}(z_0) \text{ (resp. } V') \longrightarrow w/b^n \in \tilde{C}(z_0) \text{ (resp. } V') \quad \text{for } n = 1, 2, \ldots.$$

It follows from the smoothness of $\tilde{C}(z_0)$ and the fact that $b > 1$ that $\tilde{C}(z_0)$ is a line $Au + Bv = 0$ passing through $w = 0$, proving (2) (ii-b).

For (2) (iii), similar arguments to those used in the proof of 2 (i) (which we omit) show that $X = cX_u$ for some c and $D = \cup_{c \in \delta} \sigma_c$ for some domain $\delta \subset \mathbb{P}^1$. $\square$

Given a pseudoconvex domain D in $\mathcal{H}$ with C^ω-smooth boundary, under the various cases of (2) of Lemma 4.2, depending on the relationship between the tori $\mathbf{T}_a$, $\mathbf{T}_b$ and ∂D, we want to show that either D is Stein or D is the appropriate type of non-Stein domain in Theorem 1.1. We proved in Lemma 4.1 that under certain hypotheses on ∂D the function $-\lambda[z,w]$ is a plurisubharmonic exhaustion function for D. The next step is to show that if ∂D hits, but does not contain, one of the tori $\mathbf{T}_a$ or $\mathbf{T}_b$, and D does not contain the other one, then D is Stein.

Lemma 4.3. *Let D be a pseudoconvex domain in $\mathcal{H}$ with $C^\omega-$smooth boundary. If $\emptyset \neq \partial D \cap \mathbf{T}_a \neq \mathbf{T}_a$ and $D \not\supset \mathbf{T}_b$, then D is Stein (and similarly if $\mathbf{T}_a$ and $\mathbf{T}_b$ are switched).*

The condition $D \not\supset \mathbf{T}_b$ separates into the following three cases:

$(C1)$ $\partial D \cap \mathbf{T}_b = \emptyset$, $(C2)$ $\emptyset \neq \partial D \cap \mathbf{T}_b \neq \mathbf{T}_b$ or $(C3)$ $\partial D \cap \mathbf{T}_b = \mathbf{T}_b$.

Before giving the proof we recall the following general result from [1]:

Let $\mathcal{D} : t \in B \to D(t) \subset M$ be a smooth variation of domains $D(t) \in M$ over $B \subset \mathbb{C}$ where M is a complex Lie group of dimension $n \geq 1$. Here $D(t)$ need not be relatively compact in M but $\partial D(t)$ is assumed to be $C^\infty-$smooth. Assume each domain $D(t)$ contains the identity element e. Let $g(t,z)$ and $\lambda(t)$ be the c-Green function and the c-Robin constant for $(D(t), e)$ associated to a Kähler metric and a positive, smooth function c on M. We have the following rigidity result:

$(\star 1)$ *Assume that the total space $\mathcal{D} = \cup_{t\in B}(t, D(t))$ is pseudoconvex in $B \times M$. If $\frac{\partial^2\lambda}{\partial t\partial\bar{t}}(0) = 0$, then $\frac{\partial g(t,z)}{\partial t}\big|_{t=0} \equiv 0$ on $D(0)$.*

Let $\psi(t,z)$ be a $C^\infty-$defining function of $\mathcal{D}$ in a neighborhood of $\partial\mathcal{D} = \cup_{t\in B}(t, \partial D(t))$. Since $\partial D(t)$ is smooth, we have $\frac{\partial\psi}{\partial z}(t,z) \neq 0$ for $(t,z) \in \partial\mathcal{D} = \{\psi(t,z) = 0\}$. We have a type of contrapositive of $(\star 1)$:

$(\star 2)$ Assume that $\mathcal{D}$ is pseudoconvex in $B \times M$. If there exists a point $z_0 \in \partial D(0)$ with

$$\frac{\partial\psi}{\partial t}(0, z_0) \neq 0, \tag{4.16}$$

then $\frac{\partial^2(-\lambda)}{\partial t\partial\bar{t}}(0) > 0$.

Proof of $(\star 2)$. For simplicity, we give the proof for M of complex dimension one; the general case is similar. We set $z_0 = x_0+iy_0$; $z = x+iy$ and $t = t_1+it_2$.

We may assume $\frac{\partial\psi}{\partial y}(0, z_0) \neq 0$. In a sufficiently small neighborhood $B_0 \times V$ of $(0, z_0)$ we can write $\partial D(t)$ in the form

$$y = y(t,x) := c_0(t) + c_1(t)(x - x_0) + c_2(t)(x - x_0)^2 + \dots$$

where $c_0(0) = y_0$. Using (4.16) we may assume $\frac{\partial\psi}{\partial t_1} \neq 0$ in $B_0 \times V$. Thus, $\frac{\partial c_0}{\partial t_1} \neq 0$ on B_0. We set $A = \frac{\partial c_0}{\partial t_1}(0) \neq 0$. It follows that

$$\begin{aligned} & y(t_1, x) - y(0, x) \\ & = (c_0(t_1) - c_0(0)) + (c_1(t_1) - c_1(0))(x - x_0) + (c_2(t_1) - c_2(0))(x - x_0)^2 + \cdots \\ & = t_1 \left([A + O(t_1)] + [A_1 + O(t_1)](x - x_0) + [A_2 + O(t_1)](x - x_0)^2 + \dots \right). \end{aligned}$$

We can find a sufficiently small interval $I_0 := [-r, r]$ on the x-axis and a sufficiently small interval $J_0 = [x_0 - r_0, x_0 + r_0]$ on the t_1-axis such that

$$|y(t_1, x) - y(0, x)| \geq \frac{A}{2}\,|t_1| \quad \text{on } I_0 \times J_0.$$

Using this estimate, it follows from the boundary behavior of the c-Green function $g(t,z)$ and standard potential-theoretic arguments that $\frac{\partial g(t,z)}{\partial t_1}\big|_{t=0} \neq 0$, and hence $\frac{\partial^2(-\lambda)}{\partial t \partial \bar{t}}(0) > 0$. □

Remark 4.2. *We give an example of a variation $\mathcal{D}$ which does not satisfy (4.16) of* ($\star$2). *Let $B = \{|t| < 1/2\}$ and*

$$\mathcal{D} = \{|t|^2 + |z|^2 < 1\} \cap (B \times \mathbb{C}_z).$$

Here $\psi(t,z) = \{1 - |t|^2 + |z|^2\}$ is a defining function of $\mathcal{D}$ and $D(t) = \{|z|^2 < 1 - |t|^2\}$ for $t \in B$. We have $\frac{\partial\psi}{\partial z}(0,z) = \bar{z} \neq 0$ on $\partial D(0)$, and $\frac{\partial\psi}{\partial t}(0,z) = \bar{t} = 0$ on $\partial D(0)$.

Proof of Lemma 4.3. We first want to show that if $-\lambda[z,w]$ is not strictly plurisubharmonic in D, then there is point $p_0 = [z_0, w_0]$ in D^* at which $-\lambda[z,w]$ is not strictly plurisubharmonic; then we show this cannot occur so that D is Stein. Let $\psi[z,w]$ be a defining function for D defined in a neighborhood of ∂D. We divide the proof of the lemma in two cases.

1^{st} *case.* Assume there exists $[z_0, 0] \in \partial D \cap \mathbf{T_a}$ with $z_0 \neq 0$ such that neither $\frac{\partial \psi}{\partial z}$ nor $\frac{\partial \psi}{\partial w}$ vanishes at $(z_0, 0)$ and assume case $(C1)$ or $(C2)$.

Using $(\star 2)$, we first prove the following fact in this 1^{st} case. Assume $(1,0) \in D \cap \mathbf{T}_a$. Then $-\lambda[z,w]$ is strictly subharmonic at $[1,0]$ in the direction $\mathbf{a} = (0,1)$, i.e.,

$$\frac{\partial^2(-\lambda)}{\partial \tau \partial \overline{\tau}}[1,\tau]|_{\tau=0} > 0.$$

To see this, we take a small disk $\delta := \{|\tau| < r\} \subset \mathbb{C}_\tau$ and consider the variation of domains

$$\mathfrak{D} : \ \tau \in \delta \to D(\tau) := D[1,\tau] \subset \mathbb{C}_Z^* \times \mathbb{C}_W^*.$$

Note that

$$D(\tau) = \left\{ \begin{array}{lll} \widetilde{D}^* \times (1, 1/\tau) & \text{if} & \tau \in \delta \setminus \{0\}; \\ \widetilde{D}_a \times \mathbb{C}_W^* & \text{if} & \tau = 0 \end{array} \right\}$$

(recall $D \cap \mathbf{T}_a = [D_a, 0]$). We let $\lambda(\tau) = \lambda[1,\tau]$ denote the c-Robin constant for $\big(D(\tau), (1,\tau)\big)$. We set $\mathfrak{D} := \cup_{\tau \in \delta}(\tau, D(\tau))$ and $\partial\mathfrak{D} = \cup_{\tau\in\delta}(\tau, \partial D(\tau))$. For $\tau \in \delta \setminus \{0\}$, we consider the automorphism

$$F_\tau : \ (z,w) \in \mathbb{C}_z^* \times \mathbb{C}_w^* \to (Z,W) = (z, \frac{w}{\tau}) \in \mathbb{C}_Z^* \times \mathbb{C}_W^*.$$

From the definition of $D(\tau)$, we have $D(\tau) = F_\tau(\widetilde{D}^*)$. We let $\psi(z,w)$ be a defining function for ∂D in $\mathcal{H}$; to avoid notational issues we also regard $\psi(z,w)$ as a defining function of $\partial \widetilde{D}$. For $\tau \in \delta \setminus \{0\}$ we set

$$\Phi(\tau, (Z,W)) := \psi(Z, \tau W)$$

which is a defining function for $\partial\mathfrak{D}|_{\delta\setminus\{0\}}$. Setting $\Phi[0,(Z,W)] := \psi(Z,0)$, we see that $\Phi[\tau,(Z,W)]$ becomes a smooth defining function for the entire set $\partial\mathfrak{D}$. We focus on the special point $(z_0, 1)$ in $\partial D(0)$. Then

$$\begin{aligned} \nabla_{(Z,W)}\Phi\big|_{(0,(z_0,1))} &= \big(\frac{\partial \Phi}{\partial Z}, \frac{\partial \Phi}{\partial W}\big)\big|_{(0,(z_0,1))} = \big(\frac{\partial \psi}{\partial z}, \frac{\partial \psi}{\partial w}\tau\big)\big|_{(0,(z_0,1))} \\ &= \big(\frac{\partial \psi}{\partial z}(z_0,0), 0\big) \neq (0,0) \quad \text{by the condition of the } 1^{st} \text{ step.} \end{aligned}$$

Similarly,

$$\begin{aligned} \frac{\partial \Phi}{\partial \tau}\big|_{(0,(z_0,1))} &= \frac{\partial \psi}{\partial w} W\big|_{(0,(z_0,1))} \\ &= \frac{\partial \psi}{\partial w}(z_0, 0) \neq 0 \quad \text{by the condition of the } 1^{st} \text{ step.} \end{aligned}$$

It follows from ($\star 2$) that $\frac{\partial^2(-\lambda)}{\partial\tau\partial\bar\tau}[1,\tau]|_{\tau=0} > 0$, as desired.

On the other hand, it is easy to see that $-\lambda[z,w]$ in D is strictly subharmonic at $[1,0]$ in any direction $\mathbf{a} = (a_1, a_2) \in \mathbb{C}^2 \setminus \{0\}$ with $\|\mathbf{a}\| = 1$ and $a_1 \neq 0$. Thus $-\lambda[z,w]$ in D is strictly plurisubharmonic at $[1,0]$. A similar argument shows that $-\lambda[z,w]$ in D is strictly plurisubharmonic at any point $[z,0] \in D \cap \mathbf{T}_a$.

In case $(C2)$, if there exists $[0,w_0] \in \partial D \cap \mathbf{T}_b$ with $w_0 \neq 0$ such that neither $\frac{\partial\psi}{\partial z}$ nor $\frac{\partial\psi}{\partial w}$ vanishes at $(0,w_0)$, a similar argument shows that $-\lambda[z,w]$ in D is strictly plurisubharmonic at any point $[0,w] \in D \cap \mathbf{T}_b$. Hence we conclude that if $-\lambda[z,w]$ is not strictly plurisubharmonic in D, there exists a point $p_0 = [z_0, w_0]$ in D^* at which $-\lambda[z,w]$ is not strictly plurisubharmonic. This fact is trivially true in case $(C1)$.

Now since $\partial D \not\supset \mathbf{T}_a$ and $\partial D \not\supset \mathbf{T}_b$, we are in case (2) (i) of Lemma 4.2. Hence we have $\partial D \cap (\mathbf{T}_a \cup \mathbf{T}_b) = \emptyset$. This contradicts $\partial D \cap \mathbf{T}_a \neq \emptyset$; thus D is Stein. $\square$

2^{nd} *case.* Assume there exists $[z_0, 0] \in \partial D \cap \mathbf{T_a}$ with $z_0 \neq 0$ such that neither $\frac{\partial\psi}{\partial z}$ nor $\frac{\partial\psi}{\partial w}$ vanishes at $(z_0, 0)$ and assume case $(C3)$.

Recall $\partial D \supset \mathbf{T}_b$ holds in case $(C3)$. Here we need the function $U[z,w]$ on $\mathcal{H}^*$ defined in 2. of section 2. Using 2 (b) of Lemma 4.1, i.e., for $[z_0, w_0] \in \partial D \setminus \mathbf{T}_b$,

$$-\lambda[z,w] \to \infty \quad \text{as } [z,w] \in D \to [z_0, w_0],$$

and property 2. (2) of $U[z,w]$ we see that

$$s[z,w] := \max\{-\lambda[z,w], U[z,w]\} \tag{4.17}$$

is a well-defined plurisubharmonic exhaustion function for D. In order to prove D is Stein, we use a result from § 14 in [3]: it suffices to show that for any $K \Subset D$ there exists a Stein domain D_K with $K \Subset D_K \subset D$. To construct D_K, we take $m > \max_{[z,w]\in K} |-\lambda[z,w]|$ and consider

$$v[z,w] := \max\{-\lambda[z,w] + 2m,\ \varepsilon U[z,w]\}$$

where $\varepsilon > 0$ is chosen sufficiently small so that $v[z,w] = -\lambda[z,w] + 2m$ on K. Again from property 2. (2) of $U[z,w]$, $v[z,w]$ is a well-defined plurisubharmonic exhaustion function for D. We take $M > 1$ sufficiently large so that

$$K \Subset D(M) := \{[z,w] \in D : v[z,w] < M\} \text{ and } \emptyset \neq \partial D(M) \cap \mathbf{T}_a \neq \mathbf{T}_a.$$

Note that $D(M) \Subset D$ so that $\mathbf{T}_b \cap D(M) = \emptyset$; also $\partial D(M)$ is piecewise smooth. We now have

$$\partial D(M) \cap \mathbf{T}_b = \emptyset \text{ and } \emptyset \neq \partial D(M) \cap \mathbf{T}_a \neq \mathbf{T}_a. \tag{4.18}$$

We consider the c-Robin function $\lambda_M[z,w]$ for $D(M)$. Although $\partial D(M)$ is not smooth, by the construction of $\lambda_M[z,w]$ and the fact that $\partial D(M) \not\supset \mathbf{T}_a, \mathbf{T}_b$, it follows that $-\lambda_M[z,w]$ is a smooth plurisubharmonic exhaustion function for $D(M)$.

Let $D(M,M') := \{[z,w] \in D(M) : -\lambda_M[z,w] < M'\}$ and take $M' > 1$ sufficiently large so that

$$D(M,M') \Supset K \text{and } \emptyset \neq \partial D(M,M') \not\supset \mathbf{T}_a.$$

Note that $D(M,M')$ is a pseudoconvex domain in $\mathcal{H}$ with smooth boundary and we have

$$\partial D(M,M') \cap \mathbf{T}_b = \emptyset \text{ and } \emptyset \neq \partial D(M,M') \not\supset \mathbf{T}_a. \tag{4.19}$$

It follows from the 1st case assuming ($C1$) that $D(M,M')$ is Stein, so that D is Stein.

3^{rd} *case.* Assume one of $\frac{\partial\psi}{\partial z}, \frac{\partial\psi}{\partial w}$ vanishes identically on $\partial D \cap \mathbf{T_a}$ and/or one of $\frac{\partial\psi}{\partial z}, \frac{\partial\psi}{\partial w}$ vanishes identically on $\partial D \cap \mathbf{T_b}$.

For simplicity, we assume $\frac{\partial\psi(z,0)}{\partial z} = 0$ for all $[z,0] \in \partial D \cap \mathbf{T_a}$; other cases are similar. In this case, using the fact that $-\lambda[z,w]$ is a plurisubharmonic exhaustion function on D from Lemma 4.1, for $M > 1$,

$$D_M = \{\lambda[z,w] < M\} \Subset D$$

is a pseudoconvex domain. For sufficiently large M, we claim that $\partial D_M \cap \mathbf{T}_a \neq \emptyset$ and ∂D_M satisfies the conditions of the 1^{st} or 2^{nd} case. To see this, we observe that $-\lambda[z,0]$ is a subharmonic exhaustion function for $D_a := D \cap \mathbf{T}_a$ and we take $z_0 \in D_a$ at which $-\lambda[z,0]$ attains its minimum value on D_a. Then $\frac{\partial\lambda}{\partial z}[z_0,0] = 0$ and we can find z' near z_0 such that $\frac{\partial\lambda}{\partial z}[z',0] \neq 0$ and $\frac{\partial\lambda}{\partial w}[z',0] \neq 0$. Using the real analyticity of $\lambda[z,w]$ in D, we can choose $M \gg 1$ so that D_M satisfies the appropriate conditions.

From the 1^{st} and 2^{nd} cases, we conclude that D_M is Stein. In this way we can find an increasing sequence of Stein domains $D_{M(n)}$ in D with $D_{M(1)} \Subset$

$D_{M(2)} \Subset \cdots$ and $\lim_{n\to\infty} D_{M(n)} = D$. Since D admits a plurisubharmonic exhaustion function, it follows from § 14 of [3] that D is Stein. $\square$

Next we deal with the case where ∂D contains one of $\mathbf{T}_a$ or $\mathbf{T}_b$ but not both.

Lemma 4.4. *Let D be a pseudoconvex domain in $\mathcal{H}$ with C^ω−smooth boundary. If (i) $\partial D \supset \mathbf{T}_a$ and (ii) $\partial D \cap \mathbf{T}_b \neq \mathbf{T}_b$, then*

(1) D is Stein or

(2) D is of type (**b**2) *in Theorem 1.1. In fact, $D = \bigcup_{c\in\delta} \sigma_c$ with $0 \in \partial\delta$ and $\infty \notin \partial\delta$.*

(and similarly if $\mathbf{T}_a$ and $\mathbf{T}_b$ are switched as well as 0 and ∞).

The condition (ii) separates into the following two cases:

$$(\tilde{C}1)\quad \emptyset \neq \partial D \cap \mathbf{T}_b \neq \mathbf{T}_b \qquad \text{or} \qquad (\tilde{C}2)\quad D \supset \mathbf{T}_b.$$

Proof. We first treat the case $(\tilde{C}1)$. We assume that D is not of type (**b**2) as in (2) and we show D is Stein. Here we need the function $U[z,w]$ on $\mathcal{H}^*$ defined in 2. of section 2. Using 2 (b) of Lemma 4.1, i.e., for $[z_0,w_0] \in \partial D \setminus \mathbf{T}_a$,

$$-\lambda[z,w] \to \infty \qquad \text{as } [z,w] \in D \to [z_0,w_0],$$

and property 2. (2) of $U[z,w]$ we see that

$$s[z,w] := \max\{-\lambda[z,w], -U[z,w]\} \tag{4.20}$$

is a well-defined plurisubharmonic exhaustion function for D. In order to prove D is Stein, we again appeal to § 14 in [3]: we show that for any $K \Subset D$ there exists a Stein domain D_K with $K \Subset D_K \subset D$. To construct D_K, we take $m > \max_{[z,w]\in K} |-\lambda[z,w]|$ and consider

$$v[z,w] := \max\{-\lambda[z,w] + 2m,\ -\varepsilon U[z,w]\}$$

where $\varepsilon > 0$ is chosen sufficiently small so that $v[z,w] = -\lambda[z,w] + 2m$ on K. Again from property 2. (2) of $U[z,w]$, $v[z,w]$ is a well-defined plurisubharmonic exhaustion function for D. We take $M > 1$ sufficiently large so that

$$K \Subset D(M) := \{[z,w] \in D : v[z,w] < M\}.$$

Note that $D(M) \Subset D$ so that $\mathbf{T}_a \cap D(M) = \emptyset$; also $\partial D(M)$ is piecewise smooth. Indeed, from the construction, we now have in case $(\tilde{C}1)$

$$\partial D(M) \cap \mathbf{T}_a = \emptyset \text{ and } \emptyset \neq \partial D(M) \cap \mathbf{T}_b \neq \mathbf{T}_b. \tag{4.21}$$

We consider the c-Robin function $\lambda_M[z,w]$ for $D(M)$. Although $\partial D(M)$ is not smooth, by the construction of $\lambda_M[z,w]$ and the fact that $\partial D(M) \not\supset \mathbf{T}_a, \mathbf{T}_b$, it follows that $-\lambda_M[z,w]$ is a smooth plurisubharmonic exhaustion function for $D(M)$.

Next we take $M' > 1$ sufficiently large so that

$$D(M,M') := \{[z,w] \in D(M) : -\lambda_M[z,w] < M'\} \Supset K.$$

Note that $D(M,M')$ is a pseudoconvex domain in $\mathcal{H}$ with smooth boundary; moreover by (4.21),

$$\partial D(M,M') \cap \mathbf{T}_a = \emptyset \text{ and } \emptyset \neq \partial D(M,M') \not\supset \mathbf{T}_b. \tag{4.22}$$

It follows from Lemma 4.3 that $D(M,M')$ is Stein, and hence so is D.

We next treat the case $(\tilde{C}2)$, so that $\partial D \supset \mathbf{T}_a$ and $D \supset \mathbf{T}_b$. In this setting we shall show that conclusion (2) in Lemma 4.4 holds.

Since $\mathbf{T}_b$ is compact in D, we can find a neighborhood V of $\mathbf{T}_b$ in D such that $\mathbf{T}_b \Subset V \Subset D$. Since $\Sigma_c := \{|w| = c|z|^\rho\}$ (or $\sigma_c := \{w = cz^\rho\}$) approaches $\mathbf{T}_b$ in $\mathcal{H}$ as $c \to \infty$, it follows that for c sufficiently large, the Levi flat hypersurface Σ_c satisfies $\Sigma_c \Subset V \Subset D$ (or the compact torus σ_c satisfies $\sigma_c \Subset V \Subset D$) . But $-\lambda[z,w]$ is a plurisubharmonic function on D (although not necessarily an exhaustion function); hence $-\lambda[z,w]$ is not strictly plurisubharmonic at any point in Σ_c (or σ_c). From Lemma 4.2, we conclude that D is given as in case (2) (ii) of that lemma.

For simplicity, we complete the argument if $\Sigma_c \Subset V \Subset D$. We claim that ρ is of case (**b2**) (ρ rational and τ rational) in Theorem 1.1 and hence D is of the form in case (2) (ii-a) of Lemma 4.2, completing our proof. For if ρ is of case (**a**) (ρ irrational) or of case (**b1**) (ρ rational and τ irrational), then from the proof of Lemma 4.2, we have (recall (α^*))

$$D^* = \bigcup_{c\in I} \Sigma_c = \bigcup_{c \in I} \{|w| = c|z|^\rho\}$$

where $I = (r,R)$ is an open interval in $(0,\infty)$ because D^* is connected. Since $D \supset \mathbf{T}_b$, $D = \cup_{c\in(r,\infty]}\Sigma_c$. However, since $\partial D \supset \mathbf{T}_a$, we must have $r = 0$. Thus $D = \mathcal{H} \setminus \mathbf{T}_a$ which contradicts the smoothness of ∂D. $\square$

Note in particular we have proved that the Nemirovskii-type domains in (2) (ii-b) of Lemma 4.2 are Stein. An entirely similar proof, which we omit, deals with the case where ∂D contains both $\mathbf{T}_a$ and $\mathbf{T}_b$.

Lemma 4.5. *Let D be a pseudoconvex domain in $\mathcal{H}$ with C^ω−smooth boundary. If $\partial D \supset \mathbf{T}_a \cup \mathbf{T}_b$, then*

(1) D is Stein or

(2) D is of type (**b**2) *in Theorem 1.1. In fact, $D = \bigcup_{c\in\delta} \sigma_c$ with $0, \infty \in \partial\delta$.*

We can now easily conclude with the proof of our main result.

Proof of Theorem 1.1. Let D be a pseudoconvex domain in $\mathcal{H}$ with C^ω−smooth boundary which is not Stein. We consider three "symmetric" cases depending on the nature of $\partial D \cap \mathbf{T}_a$ or $\partial D \cap \mathbf{T}_b$.

1^{st} *case:* $\partial D \supset \mathbf{T}_a$ (or $\partial D \supset \mathbf{T}_b$).

If $\partial D \supset \mathbf{T}_a$, we can have either $\partial D \cap \mathbf{T}_b \neq \mathbf{T}_b$ or $\partial D \supset \mathbf{T}_b$. If $\partial D \cap \mathbf{T}_b \neq \mathbf{T}_b$, from Lemma 4.4, $D = \bigcup_{c\in\delta} \sigma_c$ with $0 \in \partial\delta$ and $\infty \notin \partial\delta$. If $\partial D \supset \mathbf{T}_b$, this means $\partial D \supset \mathbf{T}_a \cup \mathbf{T}_b$; hence Lemma 4.5 implies $D = \bigcup_{c\in\delta} \sigma_c$ with $0, \infty \in \partial\delta$.

2^{nd} *case:* $\partial D \cap \mathbf{T}_a = \emptyset$ (or $\partial D \cap \mathbf{T}_b = \emptyset$).

If $\partial D \cap \mathbf{T}_a = \emptyset$, we can have either $\partial D \cap \mathbf{T}_b \neq \mathbf{T}_b$ or $\partial D \supset \mathbf{T}_b$. If $\partial D \supset \mathbf{T}_b$, we are done by the 1^{st} case. If $\partial D \cap \mathbf{T}_b \neq \mathbf{T}_b$, either

$$(I)\ \partial D \cap \mathbf{T}_b = \emptyset \text{ or } (II)\ \emptyset \neq \partial D \cap \mathbf{T}_b \neq \mathbf{T}_b.$$

Note that if $\partial D \cap \mathbf{T}_b = \emptyset$, then $\partial D \cap (\mathbf{T}_a \cup \mathbf{T}_b) = \emptyset$.

Let $\lambda[z,w]$ be the c-Robin function of D. From Lemma 4.1 we know that $-\lambda[z,w]$ is a plurisubharmonic exhaustion function on D. We shall prove that we can find a point $[z_0, w_0]$ in D^* at which $-\lambda[z,w]$ is not strictly plurisubharmonic. We give the proof when ρ is irrational since the other cases are completely analogous.

In this setting we have three possible situations for D: (i) $D \cap (\mathbf{T}_a \cup \mathbf{T}_b) = \emptyset$; (ii) $D \cap \mathbf{T}_a = \emptyset$ and $D \supset \mathbf{T}_b$ (or the symmetric case with $\mathbf{T}_a, \mathbf{T}_b$ switched); and (iii) $D \supset \mathbf{T}_a \cup \mathbf{T}_b$. In case (i) we are done since $D = D^*$ so that, by the assumption D is not Stein, there is a point $[z_0, w_0]$ in $D = D^*$ at which $-\lambda[z,w]$ is not strictly plurisubharmonic. By (2) (i) of Lemma 4.2, D is of type (**a**1). In case (ii), since $\mathbf{T}_b$ is compact in D, we can find a neighborhood

V of $\mathbf{T}_b$ in D such that $\mathbf{T}_b \Subset V \Subset D$. The Levi flat hypersurface $\Sigma_c := \{|w| = c|z|^{\rho}\}$ approaches $\mathbf{T}_b$ as $c \to \infty$; hence $\Sigma_c \Subset V \Subset D$ for c sufficiently large. Since $-\lambda[z, w]$ is a plurisubharmonic function on D, $-\lambda[z, w]$ is not strictly plurisubharmonic at points of Σ_c; thus we can find such a point in D^*. The conclusion of Theorem 1.1 in the case where $\partial D \cap (\mathbf{T}_a \cup \mathbf{T}_b) = \emptyset$ now follows from (2) (ii-a) of Lemma 4.2. In case (iii), similar reasoning as in case (ii) shows that $D \supset \Sigma_{c_0}$ for some $c_0 \neq 0, \infty$. It follows that $D = \bigcup_{c \in I} \Sigma_c$ where I is an interval in $[0, \infty]$. Since $D \supset \mathbf{T}_a \cup \mathbf{T}_b$, we have $I = [0, \infty]$, i.e., $D = \mathcal{H}$, which is absurd. This finishes the proof of case (I).

In this 2^{nd} case, where $\partial D \cap \mathbf{T}_a = \emptyset$, it remains to deal with case (II), i.e., $\partial D \cap \mathbf{T}_a = \emptyset$ and $\emptyset \neq \partial D \cap \mathbf{T}_b \neq \mathbf{T}_b$. We separate $\partial D \cap \mathbf{T}_a = \emptyset$ into two cases:

$$(c1)\ D \supset \mathbf{T}_a \text{ and } (c2)\ D \not\supset \mathbf{T}_a.$$

In case (c1), using the argument in case (ii) above we can find a neighborhood V of $\mathbf{T}_a$ in D such that $\mathbf{T}_b \Subset V \Subset D$ and hence $\Sigma_c \Subset V \Subset D$ for $c > 0$ sufficiently close to 0. Thus we obtain points in D^* at which $-\lambda[z, w]$ is not strictly plurisubharmonic. We now appeal to case (2) (i) of Lemma 4.2.

Finally, case (c2), cannot occur, since the assumptions $\emptyset \neq \partial D \cap \mathbf{T}_b \neq \mathbf{T}_b$ and $D \not\supset \mathbf{T}_a$ imply from Lemma 4.3 that D is Stein.

3^{rd} *case:* $\emptyset \neq \partial D \cap \mathbf{T}_a \neq \mathbf{T}_a$ (or $\emptyset \neq \partial D \cap \mathbf{T}_b \neq \mathbf{T}_b$).

If $\emptyset \neq \partial D \cap \mathbf{T}_a \neq \mathbf{T}_a$, from Lemma 4.3 we must have $D \supset \mathbf{T}_b$. Thus $\partial D \cap \mathbf{T}_b = \emptyset$ and we are done by the 2^{nd} case.

This completes the proof of Theorem 1.1. □

5 Appendix A: Proof of Lemma 3.1

We give the proof of Lemma 3.1. Assertion 1. follows from property (3) of $U[z, w]$ in Ueda's remark. To see this, X_u has integral curve $\{w = z^{\rho}\}$. Since ρ is a real number, we have $|w| = |z|^{\rho}$ on the integral curve; this is the same as

$$U[z, w] = \frac{\log |z|}{\log |a|} - \frac{\log |w|}{\log |b|} = 0.$$

Since $U[z, w]$ is a pluriharmonic exhaustion function for $\mathcal{H}^*$, $\widetilde{\Sigma} := \{U[z, w] = 0\}$ in $\mathcal{H}$ is a real three-dimensional Levi-flat closed hypersurface in $\mathcal{H}^*$. Thus

$\tilde{\Sigma}_u \subset \widetilde{\Sigma}$. Conversely, fix $z' = |z'|e^{i\theta'} \in \mathbb{C}^*$ where $0 \le \theta' < 2\pi$. We analytically continue

$$w = z^\rho = e^{\rho(\log|z| + i \arg z)}$$

starting at the point z' with initial value on the branch of $w_0' = (z')^\rho$ such that

$$w_0' = |z'|^\rho e^{i\rho\theta'}.$$

Since ρ is a real number, over z' the points in $\mathbb{C}_w$ are of the form

$$w_n' := w_0' \, e^{i2\pi n\rho}, \quad n \in \mathbb{Z}.$$

Thus $(z', w_n') \in \tilde{\sigma}_u$ for $n \in \mathbb{Z}$. Assuming ρ is irrational, we have $\{(z', w) \in \mathbb{C}^* \times \mathbb{C}^* : |w| = |z'|^\rho\} \subset \tilde{\Sigma}_u$. It follows that $\{(z, w) \in \mathbb{C}^* \times \mathbb{C}^* : |z| = |z'|, |w| = |z'|^\rho\} \subset \Sigma_u$ for any $z' \in \mathbb{C}^*$, and hence $\widetilde{\Sigma} \subset \tilde{\Sigma}_u$. This proves Assertion 1.(1) if ρ is irrational.

We next prove 1.(1) assuming τ is irrational. We have

$$\begin{aligned}
\sigma_u =& \{w = kz^{q/p}\}/\sim \\
&= \cup_{n\in\mathbb{Z}}\{(a^n z, k(a^n z)^{q/p} : z \in \mathbb{C}^*\}/\sim \\
&= \cup_{n\in\mathbb{Z}}\{(z, kb^{-n}((a^n z)^{q/p}) : z \in \mathbb{C}^*\}/\sim \\
&= \cup_{n\in\mathbb{Z}}\{(z, kz^{q/p}(a^{q/p}/b)^n) : z \in \mathbb{C}^*\}/\sim \\
&= \cup_{n\in\mathbb{Z}}\{(z, kz^{q/p}e^{i\,n2\pi\tau}) : z \in \mathbb{C}^*\}/\sim \quad (\text{since } \rho = q/p = \tfrac{\log|b|}{\log|a|}).
\end{aligned}$$

Since τ is irrational, we have $\Sigma_u = \{|w| = |z|^\rho\}$, i.e., $U[z, w] = 0$ on Σ_u, finishing the proof of 1.(1).

We next prove 1.(2). We have

$$\begin{aligned}
\sigma_u =& \{w = kz^{q/p}\}/\sim \\
&= \cup_{n\in\mathbb{Z}}\{(z, kz^{q/p}e^{i\,n2\pi\tau}) : z \in \mathbb{C}^*\}/\sim \\
&= \cup_{n\in\mathbb{Z}}\{(z, kz^{q/p}e^{i\,n2\pi m/l}) : z \in \mathbb{C}^*\}/\sim .
\end{aligned}$$

The points

$$(a^n z, k(a^n z)^{q/p}), \quad n = 1, \ldots, l-1$$

in $\mathcal{H}^*$ are distinct, while $(a^l z, k(a^l z)^{q/p})$ coincides with $(z, kz^{q/p})$. To verify this last claim, we observe that

$$\begin{aligned}(a^l z, k(a^l z)^{q/p}) &\sim (z, k(a^l z)^{q/p})/b^l \\ &= (z, kz^{q/p}e^{il((q/p)\arg a - \arg b)}) \\ &= (z, kz^{q/p}e^{il\tau}) \\ &= (z, kz^{q/p}e^{i2\pi m}) = (z, kz^{q/p}).\end{aligned}$$

Thus $w = kz^{q/p}$ defines a single-valued function on T_{a^l}, so that $\sigma_u \approx T_{a^{l/p}}$ as Riemann surfaces. We note $T_{a^{l/p}} \approx T_{b^{l/q}}$ by the definition of l.

We now prove 2.(1). Let $X = \alpha z\frac{\partial}{\partial z} + \beta w\frac{\partial}{\partial w} \notin \{cX_u : c \in \mathbb{C}\}$ with $\alpha, \beta \neq 0$. Then the integral curve $\sigma = \{\exp tX : t \in \mathbb{C}\}$ of X with initial value $e = (1,1)$ is $w = z^{\beta/\alpha}$. Let $\beta/\alpha = A + Bi$ where A, B are real. Then

$$w = z^{A+Bi} = e^{(A+Bi)\log z}.$$

Fix $z' \in \mathbb{C}^*$ and let $\operatorname{Log} z' = \log|z'| + i\theta'$ $(0 \leq \theta' < 2\pi)$ be the principal value. By analytic continuation, over z' we have

$$\begin{aligned}w_n(z') &= e^{(A+Bi)(\operatorname{Log}|z'|+i(\theta'+2n\pi))} \\ &= e^{A(\operatorname{Log}|z'|+i\theta')}\, e^{[-B(\theta'+2n\pi)]}e^{i[A2n\pi+B\operatorname{Log}|z'|]}, \quad n \in \mathbb{Z}.\end{aligned}$$

We assume $B \neq 0$, e.g., $B > 0$. Then

$$|w_n(z')| = (|z'|^A e^{-B\theta'})\, e^{-2nB\pi}, \quad n \in \mathbb{Z}.$$

Hence $\lim_{n\to\infty} |w_n(z')| = 0$ and $(z', 0) \in \Sigma$, the closure of σ in $\mathcal{H}$. Since $z' \in \mathbb{C}^*$ is arbitrary, we have $\mathbf{T}_a \subset \Sigma$.

Since $w = z^{A+Bi}$ can be written as

$$z = w^{A'+iB'} \quad \text{where } A' = \frac{A}{A^2+B^2},\ B' = -\frac{B}{A^2+B^2} < 0,$$

by analytic continuation of $z = w^{A'+iB'}$ over a given point $w' \in \mathbb{C}^*$ we have

$$\begin{aligned}z_n(w') &= e^{(A'+B'i)(\operatorname{Log}|w'|+i(\varphi'+2n\pi))} \\ &= e^{A'(\operatorname{Log}|w'|+i\varphi')}\, e^{[-B'(\varphi'+2n\pi)]}e^{i[A'(\varphi'+2n\pi)+B'\operatorname{Log}|w'|]}, \quad n \in \mathbb{Z}.\end{aligned}$$

Thus

$$\lim_{n\to\infty} |z_n(w')| = \lim_{n\to\infty} (|w'|^{A'}\ e^{-B'\varphi'})\, e^{-2nB'} = 0$$

and $(0, w') \in \Sigma$. Since $w' \in \mathbb{C}^*$ is arbitrary, we have $\mathbf{T}_b \subset \Sigma$. This proves 2.(1) in case $B \neq 0$.

In case $B = 0$ and $A \neq \rho$,

$$w = z^A = e^{A\log z} = e^{A(\log|z|+i(\theta+2n))} = |z|^A e^{iA\theta} e^{iA2n}.$$

Thus, $(z, z^A) \in s$ implies $(z, z^A e^{iA2n}) \in \sigma$, $n \in \mathbb{Z}$. By analytic continuation of z^A, we have $(a^k z, (a^k z)^A) \in \sigma$, $k \in \mathbb{Z}$.

Suppose $-\infty < A < \rho$. By analytic continuation of $w(z) = z^A = e^{A(\log|z|+i\arg z)} = |z|^A e^{iA\arg z}$ along an arbitrary path l from z to $a^k z$ where $k \in \mathbb{Z}$ is arbitrary, we have

$$w(a^k z) = (a^k z)^A = |a^k z|^A e^{iA\arg a^k z}.$$

Thus

$$p_k := (a^k z, w(a^k z)) = (a^k z, |a^k z|^A e^{iA\arg a^k z}) \in \sigma,$$

where $\arg a^k z$ takes all values $\arg a^k z + 2n\pi$ (since the path is arbitrary). In $\mathcal{H}^*$ the point p_k coincides with

$$(z, |a^k z|^A e^{iA\arg a^k z}/b^k) =: (z, \widetilde{w}_k(z)) \in \sigma.$$

Fix $z \in \mathbb{C}^*$. Using $\rho = \frac{\log|b|}{\log|a|}$,

$$|\widetilde{w}_k(z)| = |z|^A\ e^{k\log|a|(A-\rho)}.$$

Since $A < \rho$, it follows that $\lim_{k\to\infty} |\widetilde{w}_k(z)| = 0$, so that $(z, 0) \in \Sigma$. Since $z \in \mathbb{C}^*$ is arbitrary, we have $\Sigma \supset \mathbf{T}_a$.

Suppose $A > \rho$, equivalently, $1/A < 1/\rho \leq 1$. The curve $\sigma : w = z^A$ coincides with $z = w^{1/A}$. By a similar argument we see that

$$((b^k w)^{1/A}, b^k w) \in \sigma, \quad k \in \mathbb{Z}.$$

In $\mathcal{H}^*$ this point coincides with

$$(\frac{(b^k w)^{1/A}}{a^k}, w) := (\widetilde{z}_k(w), w) \in \sigma.$$

Fix $w \in \mathbb{C}^*$. Then

$$|\widetilde{z}_k(w)| = |w|^{1/A} \frac{|b|^{k/A}}{|a|^k} = |w|^{1/A} e^{k \log |b|(1/A - 1/\rho)}, \quad k \in \mathbb{Z}$$

so that $\lim_{k \to \to \infty} |\widetilde{z}_k(w)| = 0$, and hence $(0, w) \in \Sigma$. Since $w \in \mathbb{C}^*$ is arbitrary, we have $\Sigma \supset \mathbf{T}_b$, which proves 2.(1).

Finally, to prove 2.(2), let $X = \alpha z \frac{\partial}{\partial z} \neq 0$. Then the integral curve σ of X passing through $(1,1)$ is given by $(e^{\alpha t}, 1)/\sim_{(a,b)}$. In the fundamental domain $\mathcal{F}$,

$$\sigma = (\{0 < |z| \leq |a|\}, 1) \cup (\{1 < |z| \leq |a|\}, 1/b) \cup (\{1 < |z| \leq |a|\}, 1/b^2) + \ldots,$$

so that

$$\Sigma = (\{|z| \leq 1\}, 1) \cup_{n=1}^{\infty} (\{1 \leq |z| \leq |a|\}, 1/b^n) \cup \mathbf{T}_a,$$

proving 2.(2). □

6 Appendix B: Proof of Lemma 3.2

We give the proof of Lemma 3.2. The lemma is local, hence we may assume from (i) and (ii) that the unit outer normal vector of the curve $\partial D(0)$ in Δ_2 is $(0, 1)$; i.e., $\partial D(0)$ is tangent to the u-axis at $w = 0$ where $w = u + iv$. Thus, we may assume that $\psi(z, w)$ has the following Taylor expansion about the origin $(z, w) = (z, (u, v)) = (0, (0, 0))$:

$$\psi(z, w) = v + p_0(z) + p_1(z)u + p_2(z)u^2 + \ldots = 0 \tag{6.1}$$

where each $p_i(z)$, $i = 0, 1, 2, \ldots$ is a C^ω-smooth real-valued function and

$$p_0(0) = 0 \quad \text{and} \quad p_1(0) = 0.$$

We may further assume that formula (6.1) holds on $(z, u) \in \Delta_1 \times (-r_2, r_2)$. Thus we write

$$D = \{v + p_0(z) + p_1(z)u + p_2(z)u^2 + \ldots < 0 : (z, w) \in \Delta_1 \times \Delta_2\};$$

$$\mathcal{S} = \partial D = \{v + p_0(z) + p_1(z)u + p_2(z)u^2 + \ldots = 0 : (z, w) \in \Delta_1 \times \Delta_2\},$$

or equivalently,

$$D: \quad v < -\big(p_0(z) + p_1(z)u + p_2(z)u^2 + \ldots\big) \quad \text{in } \Delta_1 \times \Delta_2, \tag{6.2}$$

and, for each $z \in \Delta_1$,

$$S(z): \quad v = -\big(p_0(z) + p_1(z)u + p_2(z)u^2 + \ldots\big) \quad \text{in } \Delta_2.$$

By condition $\boxed{(iii)}$ we have

$$p_0(z) \not\equiv 0 \qquad \text{on } \Delta_1. \tag{6.3}$$

Since $\psi(z,w)$ satisfies the Levi condition (3.2) on $\psi(z,w) = 0$, using the notation

$$\psi(z,w) = \frac{w-\overline{w}}{2i} + p_0(z) + p_1(z)\frac{w+\overline{w}}{2} + p_2(z)(\frac{w+\overline{w}}{2})^2 + \ldots,$$

we calculate

$$\begin{aligned}
\frac{\partial\psi}{\partial w} &= \frac{1}{2i} + \frac{1}{2}p_1(z) + p_2(z)u + \ldots \\
\frac{\partial^2\psi}{\partial w\partial\overline{w}} &= \frac{1}{2}p_2(z) + 3p_3(z)u + \ldots \\
\frac{\partial\psi}{\partial z} &= \frac{\partial p_0(z)}{\partial z} + \frac{\partial p_1(z)}{\partial z}u + \frac{\partial p_2(z)}{\partial z}u^2 + \ldots \\
\frac{\partial\psi}{\partial z\partial\overline{z}} &= \frac{\partial^2 p_0(z)}{\partial z\partial\overline{z}} + \frac{\partial^2 p_1(z)}{\partial z\partial\overline{z}}u + \frac{\partial p_2(z)}{\partial z\partial\overline{z}}u^2 + \ldots \\
\frac{\partial^2\psi}{\partial z\partial\overline{w}} &= \frac{1}{2}\frac{\partial p_1(z)}{\partial z} + \frac{\partial p_2(z)}{\partial\overline{z}}u + \ldots
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathcal{L}\psi(z,w) = &\left(\frac{\partial^2 p_0(z)}{\partial z\partial\overline{z}} + \frac{\partial^2 p_1(z)}{\partial z\partial\overline{z}}u + \frac{\partial p_2(z)}{\partial z\partial\overline{z}}u^2 + \ldots\right)\left|\frac{1}{2i} + \frac{1}{2}p_1(z) + p_2(z)u + \ldots\right|^2 \\
&- 2\Re\left\{\left(\frac{1}{2}\frac{\partial p_1(z)}{\partial z} + \frac{\partial p_2(z)}{\partial\overline{z}}u + \ldots\right)\left(\frac{\partial p_0(z)}{\partial\overline{z}} + \frac{\partial p_1(z)}{\partial z}u + \frac{\partial p_2(z)}{\partial\overline{z}}u^2 + \ldots\right)\right. \\
&\left.\times\left(\frac{1}{2i} + \frac{1}{2}p_1(z) + p_2(z)u + \ldots\right)\right\} \\
&+ \left(\frac{1}{2}p_2(z) + 3p_3(z)u + \ldots\right)\left|\frac{\partial p_0(z)}{\partial z} + \frac{\partial p_1(z)}{\partial z}u + \frac{\partial p_2(z)}{\partial z}u^2 + \ldots\right|^2 \geq 0
\end{aligned}$$

$$\text{on } \psi(z, u+iv) = 0.$$

In particular,

$$\begin{aligned}
&\mathcal{L}\psi(z, 0 + iv) \\
&= \frac{1}{4}(1 + p_1(z)^2)\, \frac{\partial^2 p_0(z)}{\partial z \partial \overline{z}} \\
&- \frac{1}{2}\Re\left\{\frac{\partial p_1(z)}{\partial z}\, \frac{\partial p_0(z)}{\partial \overline{z}}\, (-i + p_1(z))\right\} + \frac{1}{2}\, p_2(z) \left|\frac{\partial p_0(z)}{\partial z}\right|^2 \geq 0
\end{aligned}$$

$$\text{on} \quad v + p_0(z) = 0 \text{ for } z \in \Delta_1.$$

Since this expression for $\mathcal{L}\psi(z, 0 + iv)$ is independent of v, we have

$$\begin{aligned}
(1 + p_1(z)^2)\, \frac{\partial^2 p_0(z)}{\partial z \partial \overline{z}} - 2\Re\left\{\frac{\partial p_1(z)}{\partial z}\, \frac{\partial p_0(z)}{\partial \overline{z}}\, (-i + p_1(z))\right\}& \\
+ 2p_2(z) \left|\frac{\partial p_0(z)}{\partial z}\right|^2 \geq 0 \qquad \text{for } z \in \Delta_1.& \qquad (6.4)
\end{aligned}$$

This formula will be used later on in the proof.

Claim: To prove the lemma, it suffices to show that for $r_1 > 0$ sufficiently small and $\delta_1 = \{|z| < r_1\}$,

$$(\diamondsuit) \qquad \textit{there exists} \;\; z^* \in \delta_1 \quad \textit{such that} \;\; p_0(z^*) > 0.$$

Indeed, if $(\diamondsuit)$ is true, consider the segment $[0, z^*]$ in δ_1 and the set

$$\mathbf{s} := \bigcup_{z \in [0, z^*]} S(z) \subset \Delta_2.$$

The arc $S(z)$ in Δ_2 varies continuously with $z \in \Delta_1$. Hence it follows from $0 \in S(0)$, $-ip_0^*(z^*) \in S(z^*)$, $-p(z^*) < 0$ and (6.2) that there exists a sufficiently small disk $\delta_2 \subset \Delta_2$ centered at $w = 0$ with $D(0) \cap \delta_2 \subset \mathbf{s}$.

Thus we turn to the proof of $(\diamondsuit)$. We have two cases, depending on whether $\frac{\partial p_0}{\partial z}(0)$ vanishes:

<u>Case (i).</u> $\dfrac{\partial p_0}{\partial z}(0) \neq 0$.

Since $p_0(0) = 0$, we have

$$p_0(x, y) = ax + by + O(|z|^2) \quad \text{near } z = 0$$

with $(a,b) \neq (0,0)$. It is clear that there exist $z^* \in \delta_1$ which satisfies $(\diamondsuit)$.

Case (ii). $\frac{\partial p_0}{\partial z}(0) = 0$.

In this case, we have the following Taylor expansion of $p_0(z)$ about $z=0$:

$$\begin{aligned}(1)\quad p_0(z) &= \Re\{a_{20}z^2\} + a_{11}z\overline{z}\\ &+ \Re\{a_{30}z^3 + a_{21}z^2\overline{z}\}\\ &+ \Re\{a_{40}z^4 + a_{31}z^3\overline{z}\} + a_{22}z^2\overline{z}^2\\ &+ \Re\{a_{50}z^5 + a_{41}z^4\overline{z} + a_{32}z^3\overline{z}^2\}\\ &+ \Re\{a_{60}z^6 + a_{51}z^5\overline{z} + a_{42}z^4\overline{z}^2\} + a_{33}z^3\overline{z}^3\\ &+ O(|z|^7) \qquad \text{near } z=0,\end{aligned}$$

where a_{ij} is, in general, a complex number for $i \neq j$; while a_{ii} is real.

1^{st} **Step**: Since $\frac{\partial p_0}{\partial z}(0) = 0$ and $p_0(0) = p_1(0) = 0$, inequality (6.4) reduces to

$$\frac{\partial^2 p_0}{\partial z \partial \overline{z}}(0) \geq 0, \qquad \text{i.e.,} \quad a_{11} \geq 0.$$

If $a_{11} > 0$, (1) implies that

$$\begin{aligned}\frac{\partial^2 p_0}{\partial z \partial \overline{z}}(z) &= a_{11} + O(|z|)\\ &\geq \frac{a_{11}}{2} > 0 \qquad \text{near } z = 0.\end{aligned}$$

Thus $p_0(z)$ is strictly subharmonic on a sufficiently small disk $\delta_1' := \{|z| < r'\} \subset \delta_1$; hence there exists z^* with $|z^*| = \frac{r'}{2}$ and $p_0(z^*) > p_0(0) = 0$, proving $(\diamondsuit)$.

If $a_{11} = 0$, then (1) becomes

$$\begin{aligned}p_0(z) &= \Re\{a_{20}z^2\} + O(|z|^3)\\ &= |z|^2\, \Re\left\{a_{20}e^{2i\theta} + O(|z|)\right\} \qquad \text{near } z=0,\end{aligned}$$

where $z = re^{i\theta}$.

If $a_{20} = |a_{20}|\, e^{i\theta_0} \neq 0$, for $z^* \in \delta_1$ of the form $z^* = r^* e^{-i\theta_0/2} \neq 0$ with r^* sufficiently small, we have

$$p_0(z^*) = (r^*)^2\left(|a_{20}| + O(|z^*|)\right) \geq (r^*)^2\,\frac{|a_{20}|}{2} > 0,$$

which proves ($\diamondsuit$).

Thus it remains to prove ($\diamondsuit$) when $a_{11} = a_{20} = 0$. Here,

$$
\begin{aligned}
(2) \quad p_0(z) = {} & \Re\{a_{30}z^3 + a_{21}z^2\overline{z}\} \\
& + \Re\{a_{40}z^4 + a_{31}z^3\overline{z}\} + a_{22}z^2\overline{z}^2 \\
& + \Re\{a_{50}z^5 + a_{41}z^4\overline{z} + a_{32}z^3\overline{z}^2\} \\
& + \Re\{a_{60}z^6 + a_{51}z^5\overline{z} + a_{42}z^4\overline{z}^2\} + a_{33}z^3\overline{z}^3 \\
& + O(|z|^7) \qquad \text{near } z = 0.
\end{aligned}
$$

2^{nd} **Step**: We rewrite the form of $p_0(z)$ as

$$
\begin{aligned}
p_0(z) &= \Re\{a_{30}z^3 + a_{21}z^2\overline{z}\} + O(|z|^4) \\
&= |z|^3\Re\{a_{30}e^{3i\theta} + a_{21}e^{i\theta}\} + O(|z|^4)
\end{aligned}
$$

where $z = |z|e^{i\theta}$.

We have two cases:

<u>Case (i): $(a_{30}, a_{21}) \neq (0,0)$.</u>

We consider the nonconstant polynomial

$$w = g(z) = a_{30}z^3 + a_{21}z \quad \text{in } \mathbb{C}_z.$$

Let $C = \{|z| = 1\} \subset \mathbb{C}_z$. Since $g(0) = 0$, it follows from the argument principle that the winding number of the image curve $g(C)$ in $\mathbb{C}_w$ about $w = 0$ is at least 1. In particular, $g(C)$ intersects the positive real axis in the w-plane. Hence there exists $\theta^* \in [0, 2\pi]$ such that $g(e^{i\theta^*}) > 0$. Thus if we take $z^* = r^*e^{i\theta^*}$ with r^* sufficiently small, then

$$p_0(z^*) = (r^*)^3(\Re\{g(e^{i\theta^*})\} + O(|z^*|)) = (r^*)^3\,\frac{g(e^{i\theta^*})}{2} > 0,$$

which proves ($\diamondsuit$) in this case.

<u>Case (ii): $(a_{30}, a_{21}) = (0,0)$.</u>

In this situation, we have

$$(3) \quad p_0(z) = \Re\{a_{40}z^4 + a_{31}z^3\overline{z}\} + a_{22}z^2\overline{z}^2 + \Re\{a_{50}z^5 + a_{41}z^4\overline{z} + a_{32}z^3\overline{z}^2\} + \Re\{a_{60}z^6 + a_{51}z^5\overline{z} + a_{42}z^4\overline{z}^2\} + a_{33}z^3\overline{z}^3 + O(|z|^7) \qquad \text{near } z = 0.$$

3^{rd} **Step**: We return to inequality (6.4) which we rewrite using the representation (3) of $p_0(z)$. Thus we calculate:

$$\frac{\partial p_0(z)}{\partial \overline{z}} = \frac{1}{2}\big(a_{31}z^3 + 4\,\overline{a_{40}}\,\overline{z}^3 + 3\,\overline{a_{31}}\,\overline{z}^2 z\big) + 2a_{22}\overline{z}z^2 + O(|z|^4);$$

$$\frac{\partial^2 p_0(z)}{\partial z \partial \overline{z}} = \Re\{3a_{31}z^2\} + 4a_{22}z\overline{z} + O(|z|^3). \tag{6.5}$$

Using $p_1(0) = 0$, we note that for z in a sufficiently small disk $\delta' := \{|z| < \rho'\}$, we have

$$p_1(z) = O(|z|), \quad \frac{\partial p_1}{\partial z}(z) = C_1 + O(|z|), \quad \boxed{\frac{\partial p_0}{\partial \overline{z}} = O(|z|^3)}.$$

Substituting in (6.4), for $z \in \delta'$,

$$(1 + O(|z|^2))\left(\Re\{3a_{31}z^2\} + 4a_{22}z\overline{z} + O(|z|^3)\right) - 2O(|z|^3) + 2O(|z|^6) \geq 0.$$

Consequently,

$$\left(\Re\{3a_{31}z^2\} + 4a_{22}z\overline{z}\right) + O(|z|^3) \geq 0 \qquad \text{for } z \in \delta'.$$

We write $z = re^{i\theta} \neq 0$ and divide both sides by $|z|^2 > 0$:

$$\Re\{3a_{31}e^{2i\theta}\} + 4a_{22} + O(|z|) \geq 0 \;\text{ for }\; z = r\,e^{i\theta} \in \delta',$$

$$\text{i.e.,}\quad a_{22} \;\geq\; -\Re\{\frac{3}{4}\,a_{31}e^{2i\theta}\} - O(|z|) \;\text{ on } \delta'. \tag{6.6}$$

We return to (3) and use (6.6): for $z = r\,e^{i\theta}$,

$$\begin{aligned}
p_0(z) &= \Re\left\{a_{40}z^4 + a_{31}z^3\overline{z}\right\} + a_{22}z^2 + O(|z|^5) \\
&= |z|^4\left(\Re\left\{a_{40}e^{4i\theta} + a_{31}e^{2i\theta}\right\} + a_{22} + O(|z|)\right) \\
&\geq |z|^4\left(\Re\left\{a_{40}e^{4i\theta} + a_{31}e^{2i\theta}\right\} + (-\Re\{\frac{3}{4}\,a_{31}e^{2i\theta}\} - O(|z|)) + O(|z|)\right) \\
&= |z|^4\left(\Re\left\{a_{40}e^{4i\theta} + \frac{1}{4}\,a_{31}e^{2i\theta}\right\} + O(|z|)\right).
\end{aligned}$$

We again consider two cases.

Case (i): $(a_{40}, a_{31}) \neq (0,0)$.

We consider the nonconstant polynomial

$$w = g(z) = a_{40}z^4 + \frac{1}{4}a_{31}z^2 \quad \text{in } \mathbb{C}_z.$$

Again let $C = \{|z| = 1\} \subset \mathbb{C}_z$. Since $g(z)$ vanishes to order at least 2 at 0, the winding number of the closed curve $g(C)$ about $w = 0$ is at least 2; hence $g(C)$ intersects the positive u-axis in $\mathbb{C}_w$. Thus there exists $0 \leq \theta^* < 2\pi$ with

$$A := \Re\left\{a_{40}e^{4i\theta^*} + \frac{1}{4}\,a_{31}e^{2i\theta^*}\right\} > 0.$$

If we choose $r^* > 0$ sufficiently small and set $z^* := r^*e^{i\theta^*}$, then

$$p_0(z^*) \geq (r^*)^4\ \left(\ A - |O(|z^*|)|\ \right) > (r^*)^4\ \frac{A}{2} > 0,$$

which proves ($\diamondsuit$) in this case.

Case (ii): $(a_{40}, a_{31}) = (0,0)$.

In this case, we let $z \to 0$ in inequality (6.6) to obtain

$$a_{22} \geq 0.$$

We divide Case (ii) into two subcases:

Case (ii)-(1) : $a_{22} > 0$.

By (6.5) we have

$$\frac{\partial^2 p_0(z)}{\partial z \partial \overline{z}} = |z|^2(4a_{22} + O(|z|)$$

and hence

$$\frac{\partial^2 p_0(z)}{\partial z \partial \overline{z}} \geq 2a_{22}\, |z|^2 > 0$$

for $|z|$ sufficiently small. As in a previous step, we conclude that $p_0(z)$ is a strictly subharmonic function near $z = 0$ and hence there exists z^* with $p_0(z^*) > p_0(0) = 0$.

<u>Case (ii)-(2): $a_{22} = 0$.</u>

In this situation we have $a_{40} = a_{31} = a_{22} = 0$ and we must prove $(\diamondsuit)$ with the following form of $p_0(z)$:

$$\begin{aligned} (4) \quad p_0(z) &= \Re\left\{a_{50}z^5 + a_{41}z^4\overline{z} + a_{32}z^3\overline{z}^2\right\} \\ &+ \Re\left\{a_{60}z^6 + a_{51}z^5\overline{z} + a_{42}z^4\overline{z}^2\right\} + a_{33}z^3\overline{z}^3 \\ &+ O(|z|^7) \qquad \text{near } z = 0. \end{aligned}$$

4^{th} **Step**: Using an argument as in the 2^{nd} step one can show that $(\diamondsuit)$ holds when $(a_{50}, a_{41}, a_{32}) \neq (0,0,0)$. It remains to prove $(\diamondsuit)$ in the following case:

$$\begin{aligned} (5) \quad p_0(z) &= \Re\left\{a_{60}z^6 + a_{51}z^5\overline{z} + a_{42}z^4\overline{z}^2\right\} + a_{33}z^3\overline{z}^3 \\ &+ O(|z|^7) \qquad \text{near } z = 0, \end{aligned}$$

5^{th} **Step**: The proof follows that of the 3^{rd} Step. We first prove that $(\diamondsuit)$ is true in all cases except when all coefficients $a_{ij} = 0$ for $i+j = 6$, $j = 0, 1, 2, 3$.

Using (5), for $z \in \delta_1 = \{|z| < \rho_1\}$ with ρ_1 sufficiently small,

$$\begin{aligned} \frac{\partial p_0(z)}{\partial z} &= O(|z|^5); \\ \frac{\partial^2 p_0(z)}{\partial z \partial \overline{z}} &= \Re\left\{5a_{51}z^4 + 8a_{42}z^3\overline{z}\right\} + 9a_{33}z^2\overline{z}^2 + O(|z|^5). \end{aligned} \tag{6.7}$$

Once again using the Levi condition (6.4), we have

$$\begin{aligned} (1 + O(|z|^2))\ &\left(\Re\left\{5a_{51}z^4 + 8a_{42}z^3\overline{z}\right\} + 9a_{33}z^2\overline{z}^2 + O(|z|^5)\right) \\ &- 2O(|z|^5) + 2O(|z|^{10}) \geq 0 \quad \text{on } \delta_1. \end{aligned}$$

Setting $z = re^{i\theta} \neq 0$, we divide by $|z|^4 > 0$ to obtain

$$\Re\left\{5a_{51}e^{4i\theta} + 8a_{42}e^{2i\theta}\right\} + 9a_{33} + O(|r|) \geq 0 \quad \text{for } z \in \delta_1.$$

$$\text{thus} \quad a_{33} \geq -\frac{1}{9}\,\Re\left\{5a_{51}e^{4i\theta} + 8a_{42}e^{2i\theta}\right\} - O(|r|) \quad \text{for } z \in \delta_1. \tag{6.8}$$

Letting $z \to 0$, we have

$$a_{33} \geq -\frac{1}{9}\,\Re\left\{5a_{51}e^{4i\theta} + 8a_{42}e^{2i\theta}\right\} \qquad \text{for } \theta \in [0, 2\pi]. \tag{6.9}$$

Substituting (6.9) in the representation (5) of $p_0(z)$, we have

$$\begin{aligned} p_0(z) &= |z|^6 \left(\Re\left\{ a_{60}e^{6i\theta} + a_{51}e^{4i\theta} + a_{42}e^{2i\theta}\right\} + \boxed{a_{33}} + O(|z|)\right) \\ &\geq |z|^6 \Bigg(\Re\left\{ a_{60}e^{6i\theta} + a_{51}e^{4i\theta} + a_{42}e^{2i\theta}\right\} \\ &\qquad\qquad -\frac{1}{9}\,\Re\left\{5a_{51}e^{4i\theta} + 8a_{42}e^{2i\theta}\right\} + O(|z|)\Bigg) \\ &= |z|^6 \left(\Re\left\{ a_{60}e^{6i\theta} + \frac{4}{9}\,a_{51}e^{4i\theta} + \frac{1}{9}a_{42}e^{2i\theta}\right\} + O(|z|)\right). \end{aligned} \tag{6.10}$$

We consider two cases.

Case (i): $(a_{60}, a_{51}, a_{42}) \neq (0, 0, 0)$.

The nonconstant polynomial

$$w = g(z) := a_{60}\, z^6 + \frac{4}{9}\,a_{51}\, z^4 + \frac{1}{9}\,a_{42}\, z^2$$

maps the unit circle C to a closed curve $g(C)$ whose winding number about $w = 0$ is at least 2. Thus, $g(C)$ intersects the positive u-axis of the w-plane, so that

$$\exists\ \theta^* \in [0, 2\pi] \quad \text{such that} \quad g(e^{i\theta^*}) > 0. \tag{6.11}$$

Setting $z^* = r^* e^{i\theta^*}$ where $0 < r^* \ll 1$, we have by (6.10),

$$\begin{aligned} p_0(z^*) &\geq (r^*)^6 \left(g(e^{i\theta^*}) + O(|z^*|)\right) \\ &> \frac{1}{2}\,(r^*)^6\, g(e^{i\theta^*}) > 0, \end{aligned}$$

which proves ($\diamondsuit$).

Case (ii): $(a_{60}, a_{51}, a_{42}) = (0, 0, 0)$.

In this case, inequality (6.9) becomes

$$a_{33} \geq 0.$$

We divide Case (ii) into two subcases:

Case (ii)-(1): $a_{33} > 0$.

In this case, the representation (6.7) of $\dfrac{\partial^2 p_0(z)}{\partial z \partial \overline{z}}$ on δ_1 becomes

$$\frac{\partial^2 p_0(z)}{\partial z \partial \overline{z}} = 9a_{33}\,|z|^4 + O(|z|^5) = |z|^4\,\left(9a_{33} + O(|z|)\right)$$

so that

$$\frac{\partial^2 p_0(z)}{\partial z \partial \overline{z}} \geq \frac{9}{2}\,a_{33}\,|z|^4 > 0$$

on $\delta_1' = \{|z| < \rho_1'\} \subset \delta_1$. Thus $p_0(z)$ is a strictly subharmonic function on $\delta_1' \setminus \{0\}$, which implies ($\diamondsuit$).

Case (ii)-(2): $a_{33} = 0$.

In this case the representation (5) of $p_0(z)$ reduces to $p_0(z) = O(|z|^7)$ on Δ_1. By continuing these steps inductively [3] we conclude that ($\diamondsuit$) is true unless $p_0(z) \equiv 0$ on Δ_1. Using (6.3) we complete the proof of ($\diamondsuit$), and hence that of Lemma 3.2. $\square$

---

[3] Assume

$$\begin{aligned} p_0(z) = \Re\left\{a_{2n-1,0}z^{2n-1} + a_{2n-2,1}z^{2n-2}\overline{z} + \cdots + a_{n,n-1}z^n\overline{z}^{n-1}\right\} \\ + O(|z|^{2n}) \qquad \text{near } z = 0. \end{aligned}$$

Then using an argument as in the 2^{nd} step one can show that ($\diamondsuit$) holds when $(a_{2n-1,0}, a_{2n-2,2}, \ldots, a_{n,n-1}) \neq (0, 0, \ldots, 0)$. It remains to prove ($\diamondsuit$) in the case

$$\begin{aligned} (6) \quad p_0(z) = \Re\left\{a_{2n,0}z^{2n} + a_{2n-1,1}z^{2n-1}\overline{z} + \cdots + a_{n+1,n-1}z^{n+1}\overline{z}^{n-1}\right\} + a_{n,n}|z|^{2n} \\ + O(|z|^{2n+1}) \qquad \text{near } z = 0. \end{aligned}$$

References

[1] K. T. Kim, N. Levenberg and H. Yamaguchi, *Robin functions for complex manifolds and applications*, Memoirs of Amer. Math. Soc., Vol. 209, No. 984, 2011.

[2] S. U. Nemirovskii, *Stein domain with Levi-Flat Boundaries on compact complex surfaces*, Mathematical Notes, **66** No. 4 (1999), 522–545.

[3] K. Oka *Sur les fonctions analytiques de plusieurs variables III. Domains pseudoconvexes*, Tohoku Math. J. **17** (1950), 15–52.

Then, for $z \in \delta_1 = \{|z| < \rho_1\}$ with ρ_1 sufficiently small,

$$\frac{\partial p_0(z)}{\partial z} = O(|z|^{2n-1});$$

$$\begin{aligned}\frac{\partial^2 p_0(z)}{\partial z \partial \overline{z}} \\ = \Re\Big\{&(2n-1)a_{2n-1,1}z^{2n-2} + (2n-2)2a_{2n-2,2}z^{2n-3}\overline{z} + \cdots + (n+1)(n-1)a_{n+1,n-1}z^n\overline{z}^{n-2}\Big\} \\ &+ n^2 a_{n,n}|z|^{2n-2} + O(|z|^{2n-1}).\end{aligned}$$

Once again using the Levi condition (6.4), by the similar argument as (6.8) we have

$$\begin{aligned}a_{n,n} \geq -\frac{1}{n^2}\,\Re\Big\{&(2n-1)a_{2n-1,1}e^{(2n-2)\theta} + (2n-2)2a_{2n-2,2}e^{(2n-4)\theta} \\ &+ \cdots + (n+1)(n-1)a_{n+1,n-1}e^{2\theta}\Big\} \quad \text{for } \theta \in [0, 2\pi].\end{aligned}$$

Substituting this in (6) and puting $z = |z|e^{i\theta}$, we have

$$\begin{aligned}p_0(z) \geq |z|^{2n}\bigg(\Re\bigg\{&a_{2n,0}e^{2n\,i\theta} + a_{2n-1,1}\left(1 - \frac{2n-1}{n^2}\right)e^{(2n-2)\,i\theta} \\ &+ a_{2n-2,2}\left(1 - \frac{(2n-2)2}{n^2}\right)e^{(2n-4)\,i\theta} + \ldots + a_{n+1,n-1}\left(1 - \frac{(n+1)(n-1)}{n^2}\right)e^{2\,i\theta}\bigg\} + O(|z|)\bigg).\end{aligned}$$

Since the number $\cdots$ in each $\left(\cdots\right)$ is not zero, we reach ($\diamondsuit$) using the same arguement as in the 5^{th} step.